Berkeley Math Circle
Monthly Contest 8
Due April 28, 2009

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 8
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Determine whether there exists a natural number having exactly 10 divisors (including itself and 1), each ending in a different digit.

2. Two bikers, Bill and Sal, simultaneously set off from one end of a straight road. Neither biker moves at a constant rate, but each continues biking until he reaches one end of the road, at which he instantaneously turns around. When they meet at the opposite end from where they started, Bill has traveled the length of the road eleven times and Sal seven times. Find the number of times the bikers passed each other moving in opposite directions.

3. A position of the hands of a (12-hour, analog) clock is called *valid* if it occurs in the course of a day. For example, the position with both hands on the 12 is valid; the position with both hands on the 6 is not. A position of the hands is called *bivalid* if it is valid and, in addition, the position formed by interchanging the hour and minute hands is valid. Find the number of bivalid positions.

4. Let ABC be a triangle, and let M and N be the respective midpoints of AB and AC. Suppose that
$$\frac{CM}{AC} = \frac{\sqrt{3}}{2}.$$
Prove that
$$\frac{BN}{AB} = \frac{\sqrt{3}}{2}.$$

5. Let M_n be the number of integers N such that
   (a) $0 \le N < 10^n$;
   (b) N is divisible by 4;
   (c) The sum of the digits of N is also divisible by 4.

   Prove that $M_n \neq 10^n/16$ for all positive integers n.