Berkeley Math Circle
Monthly Contest 7 – Solutions

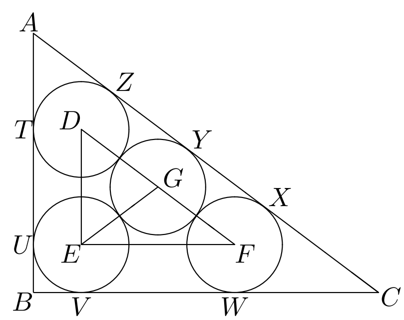
1. In the sequence
$$77492836181624186886128\ldots,$$
all of the digits except the first two are obtained by writing down the products of pairs of consecutive digits. Prove that infinitely many 6s appear in the sequence.

Solution. Since 868 appears in the sequence and $8 \cdot 6 = 6 \cdot 8 = 48$, 4848 appears later in the sequence. Then $4 \cdot 8 = 32$, so 3232 appears somewhere later. And so on:
$$868 \to 4848 \to 3232 \to 666 \to 3636 \to 1818 \to 888 \to 6464 \to 2424 \to 888 \cdots$$
It is clear that the three last sequences repeat indefinitely. Therefore 6464 appears infinitely many times, and in particular 6 appears infinitely many times.

2. Let k be a rational number greater than 1 (correction by Fengning Ding). Prove that there exist positive integers a, b, c satisfying the equations
$$a^2 + b^2 = c^2 \tag{1}$$
$$\frac{a+c}{b} = k. \tag{2}$$

Solution. Let $k = x/y$, where x and y are positive integers. We find that $x > y$. It suffices to note that
$$a = x^2 - y^2, \quad b = 2xy, \quad c = x^2 + y^2$$
are positive integers satisfying both of the given equations.

3. Four congruent circles are tangent to each other and to the sides of a triangle as shown.



(a) Prove that $\angle ABC = 90°$.

(b) If $AB = 3$ and $BC = 4$, find the radius of the circles.

Solution. Denote the centers of the four circles by D, E, F, G and their tangency points with the sides of $\triangle ABC$ by T, U, V, W, X, Y, Z as shown in the diagram. Segments TD and UE are congruent (because they are radii of congruent circles) and parallel (because they are perpendicular to AB). Therefore $DTUE$ is a parallelogram, in fact a rectangle since $\angle DTU = 90°$. Similarly, quadrilaterals $EVWF$, $FXYG$, and $GYZD$ are rectangles. We find that D, G, and F are collinear; we also have $\triangle DEF \sim \triangle ABC$, since corresponding sides are parallel.

For part (a), we note that $DG = EG = FG$, so $\angle DEF$ is inscribed in a semicircle centered at G. If $\angle DEF$ is right, so is $\angle ABC$.

For part (b), let r be the desired radius. Noting $AC = 5$ by Pythagoras, we have
$$DF = 4r, \quad DE = \frac{3}{5} \cdot 4r = \frac{12r}{5}, \quad EF = \frac{4}{5} \cdot 4r = \frac{16r}{5}.$$
We now compute $AB + BC - AC = 3 + 4 - 5 = 2$ in another way:
$$\begin{aligned} AB + BC - AC &= (AT + DE + UB) + (BV + EF + WC) - (CX + FD + ZA) \\ &= (AT - AZ) + (WC - XC) + DE + UB + BV + EF - FD \\ &= 0 + 0 + \frac{12r}{5} + r + r + \frac{16r}{5} - 4r \\ &= \frac{(12 + 5 + 5 + 16 - 20)r}{5} = \frac{18r}{5}. \end{aligned}$$

Thus $r = 2 \cdot 5/18 = 5/9$.

4. Find all pairs (a, b) of positive integers such that
$$1 + 5^a = 6^b.$$

Solution *(Based on work by Fengning Ding)* The only solution is $(1, 1)$.

It is clear that if $b = 1$ then $a = 1$, and that $(1, 1)$ is a solution. Consequently assume $b > 1$. Then 6^b is divisible by 4. On the other hand, since $5^a \equiv 1^a = 1 \bmod 4$ for all a, the left side is $2 \bmod 4$. Thus there are no solutions for $b > 1$.

5. The *tower function of twos,* $T(n)$, is defined by $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Prove that $T(n) - T(n-1)$ is divisible by $n!$ for $n \geq 2$.

Solution. Let $U(n) = T(n) - T(n-1)$. Note that
$$U(n) = T(n) - T(n-1) = 2^{T(n-1)} - 2^{T(n-2)} = 2^{T(n-2)}\left(2^{T(n-1)-T(n-2)} - 1\right) = T(n-1)\left(2^{U(n-1)} - 1\right).$$

We first prove two lemmas, the second a refinement of the first.

Lemma 1. $U(n-1) \mid U(n)$.

Proof. By induction on n. It is clear when $n = 2$ and $n = 3$. Suppose that $U(n-2) \mid U(n-1)$, $n \geq 3$. Then
$$\frac{U(n)}{U(n-1)} = \frac{T(n-1)}{T(n-2)} \cdot \frac{2^{U(n-1)} - 1}{2^{U(n-2)} - 1}.$$

The first fraction is clearly an integer. Since $a \mid b$ implies $2^a - 1 \mid 2^b - 1$, the second fraction is an integer as well. □

Lemma 2. *If p is a prime and $U(k)$ is divisible by p^r, $r \geq 1$, then $U(k+1)$ is divisible by p^{r+1}.*

Proof. If $p = 2$, the r factors of 2 in $U(k)$ must be contained in $T(k-1)$. Since $T(k) > T(k-1)$, $T(k)$ has at least $r + 1$ factors of 2, and so does $U(k+1)$.

If $p > 2$, we use induction on n. Let n_0 be the smallest value such that $p \mid U(n_0)$. Let r_0 be the number of factors of p in $U(n_0)$. Since $T(n_0 - 1)$ is a power of 2, the r_0 factors are all contained in $2^{U(n_0-1)} - 1$, i.e.
$$2^{U(n_0-1)} \equiv 1 \bmod p^{r_0}.$$

By the preceding lemma, $U(n_0)$ is divisible by $U(n_0 - 1)$; however, by definition of n_0, $U(n_0)$ is divisible by p while $U(n_0 - 1)$ is not. Consequently
$$U(n_0) = U(n_0 - 1) \cdot p \cdot J, \tag{3}$$
where J is an integer. We find that
$$\begin{aligned} 2^{U(n_0-1)} &\equiv 1 \bmod p^{r_0} \\ 2^{U(n_0-1)\cdot J} &\equiv 1 \bmod p^{r_0} \\ 2^{U(n_0)} = 2^{U(n_0-1)\cdot J\cdot p} &\equiv 1 \bmod p^{r_0+1}, \end{aligned}$$

where we have used the fact that $a \equiv b \bmod p^r$ implies $a^p \equiv b^p \bmod p^{r+1}$. We conclude that $p^{r_0+1} \mid 2^{U(n_0)} - 1 \mid U(n_0 + 1)$. This completes the base case. The induction step is similar, except that n_0 and r_0 are replaced by n and r, and equation (3) follows from the induction hypothesis rather than the definition of n_0. □

We prove the following statement by strong induction for $n \geq 2$:

Lemma 3. $U(n)$ *is divisible by all prime powers less than* $3 \cdot 2^{n-2} - 1$.

Proof. We can verify the statement by hand for $n = 2$ and $n = 3$. Thus we suppose that $U(k) = T(k) - T(k-1)$ is divisible by $k!$ for $k < n$, where $n \geq 4$. We prove the second assertion first. Let p^r be a prime power less than $3 \cdot 2^{n-2} - 1$. If $r > 1$, we note that

$$p^{r-1} \leq \frac{p^r}{2} \leq \frac{3 \cdot 2^{n-2} - 2}{2} = 3 \cdot 2^{n-3} - 1;$$

furthermore, at least one of these inequalities is strict, since the first becomes equality only if $p = 2$, and $2^n - 1 < 3 \cdot 2^{n-2} < 2^n$ when $n \geq 4$. Thus by the induction hypothesis, p^{r-1} divides U_{n-1}, and we are done by Lemma 2. Now we assume that $r = 1$. We may assume that $p \geq 5$ since U_3 is already divisible by 2 and 3. Then $p - 1$ is composite, and we can resolve it into prime powers

$$p - 1 = q_1^{a_1} q_2^{a_2} \cdots q_s^{a_s}$$

where each factor $q_i^{a_i}$ is at most

$$\frac{p-1}{2} < \frac{3 \cdot 2^{n-2} - 2}{2} = 3 \cdot 2^{n-3} - 1.$$

We infer that each $q_i^{a_i}$ divides U_{n-1}, and therefore $p - 1$ divides U_{n-1}. We now have

$$p \mid 2^{p-1} - 1 \mid 2^{U_{n-1}} - 1 \mid U_n,$$

as desired. $\square$

We are now ready to solve the problem. Assume that $n \geq 4$; the cases $n = 2$ and $n = 3$ are easily checked by hand. Let p be a prime dividing $n!$, that is, $p \leq n$. The number of factors of p in $n!$ is

$$R = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{n}{p^i} = n \sum_{i=1}^{\infty} \frac{1}{p^i} = n \cdot \frac{1/p}{1 - 1/p} = \frac{n}{p-1},$$

so

$$R \leq \frac{n-1}{p-1}.$$

We observe that $p \mid U(p)$: by hand calculation if $p = 2$, and because $p < 3 \cdot 2^{p-2} - 1$ if $p > 3$. By Lemma 2, applied $n - p$ times,

$$p^{n-p+1} \mid U(n).$$

Since $n - p + 1 \geq \frac{n-1}{p-1}$ if $n \geq p$, we have $p^R \mid U(n)$, as desired.