Berkeley Math Circle
Monthly Contest 5 – Solutions

1. In how many ways can each square of a 2×9 board be colored red, blue, or green so that no two squares that share an edge are the same color? a 100×100 grid is colored black or white so that there is at least one square of each color. Prove that there is a point with is a vertex of exactly one black square.

   Solution. Orient the board so that it consists of 9 rows of 2 squares each. The upper left square can be colored in 3 ways; the square to its right, in 2 ways. We continue from top to bottom, coloring one row at a time. If X and Y denote the (obviously different) colors of one row and Z the color not appearing in that row, the possibilities of the row below are limited to YX, YZ, and ZX. Thus there are 3 ways to color each of the eight rows below the first. The total number of colorings of the board is

   $$(3 \cdot 2) \cdot 3^8 = 39366.$$

2. Prove that the sum of the 2009th powers of the first 2009 positive integers is divisible by 2009.

   Solution. Using the factoring formula

   $$x^{2009} + y^{2009} = (x+y)(x^{2008} - x^{2007}y + x^{2006}y^2 - \cdots - xy^{2007} + y^{2008},$$

   we find that each of the numbers

   $$0^{2009} + 2009^{2009}, 1^{2009} + 2008^{2009}, 2^{2009} + 2007^{2009}, \ldots, 1004^{2009} + 1005^{2009}$$

   is divisible by 2009. Consequently so is their sum.

3. If real numbers a, b, c, d satisfy

   $$\frac{a+b}{c+d} = \frac{b+c}{a+d} \neq -1,$$

   prove that $a = c$.

   Solution. Assume for the sake of contradiction that $a \neq c$. Cross-multiplying,

   $$\begin{aligned}
   (a+b)(a+d) &= (c+b)(c+d) \\
   a^2 + ab + ad + bd &= c^2 + bc + cd + bd \\
   a^2 - c^2 + ab - bc + ad - cd &= 0 \\
   (a-c)(a+c) + (a-c)b + (a-c)d &= 0 \\
   a + c + b + d &= 0 \\
   b + c &= -a - d.
   \end{aligned}$$

   Because we are given that $a + d \neq 0$, we obtain

   $$\frac{b+c}{a+d} = -1,$$

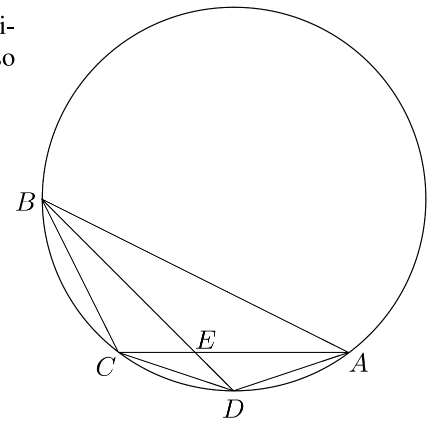
   a contradiction.

4. In triangle ABC, the bisector of $\angle B$ meets the circumcircle of $\triangle ABC$ at D. Prove that

   $$BD^2 > BA \cdot BC.$$

Solution: In the diagram, $\angle ABD = \angle DBC$ (angle bisector) and $\angle BAC = \angle BDC$ (both intercept arc BC), so $\triangle BAE \sim \triangle BDC$. We get

$$\frac{BA}{BE} = \frac{BD}{BC}$$
$$BD \cdot BE = BA \cdot BC,$$



from which the desired inequality follows since $BE < BD$.

5. A calculator has a switch and four buttons with the following functions:

- Flipping the switch from the down to the up position adds 1 to the number in the display.
- Flipping the switch from the up to the down position subtracts 1 from the number in the display.
- Pressing the red button multiplies the number in the display by 3.
- If the number in the display is divisible by 3, pressing the yellow button divides it by 3; otherwise the yellow button is nonfunctional.
- Pressing the green button multiplies the number in the display by 5.
- If the number in the display is divisible by 5, pressing the blue button divides it by 5; otherwise the blue button is nonfunctional.

Solution. Assume that not every positive integer can appear in the display and let N be the smallest number that cannot. We first note that since the buttons do not change the parity of the display, it must be even when the switch is down and odd when the switch is up. Consequently N is even, for if it were odd we could obtain N from the displayable number $N-1$ by flipping the switch. N cannot be divisible by 3, for otherwise we could get it from the displayable number $\frac{N}{3}$. If N were $2 \mod 3$, then multiplying by 3 and flipping the switch would suffice to create N from the smaller integer $\frac{N+1}{3}$. We conclude that $N \equiv 1 \mod 3$, so $N = 6n + 4$ for some integer n. Because

$$6n + 4 = \frac{3(10n+7) - 1}{5},$$

neither $10n + 7$ nor $10n + 6$, which differs from it only in the position of the switch, is obtainable. Similarly, since

$$6n + 4 = \frac{3(10n+8) + 1}{5} - 1,$$

$10n + 8$ is not obtainable. Of the three consecutive integers $10n + 6$, $10n + 7$, and $10n + 8$, at least one is a multiple of 3. Since one-third of that number is necessarily less than $6n + 4$, it can be displayed, and we have a contradiction.
Remark. In this solution it is noticeable that the green button is never used.