Berkeley Math Circle
Monthly Contest 5
Due February 3, 2009

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 5
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. In how many ways can each square of a 2×9 board be colored red, blue, or green so that no two squares that share an edge are the same color?

2. Prove that the sum of the 2009th powers of the first 2009 positive integers is divisible by 2009.

3. If real numbers a, b, c, d satisfy
$$\frac{a+b}{c+d} = \frac{b+c}{a+d} \neq -1,$$
prove that $a = c$.

4. In triangle ABC, the bisector of $\angle B$ meets the circumcircle of $\triangle ABC$ at D. Prove that
$$BD^2 > BA \cdot BC.$$

5. A calculator has a switch and four buttons with the following functions:
   - Flipping the switch from the down to the up position adds 1 to the number in the display.
   - Flipping the switch from the up to the down position subtracts 1 from the number in the display.
   - Pressing the red button multiplies the number in the display by 3.
   - If the number in the display is divisible by 3, pressing the yellow button divides it by 3; otherwise the yellow button is nonfunctional.
   - Pressing the green button multiplies the number in the display by 5.
   - If the number in the display is divisible by 5, pressing the blue button divides it by 5; otherwise the blue button is nonfunctional.

   Initially the display shows 0 and the switch is down. Prove that we can get any positive integer we wish in the display.