Berkeley Math Circle
Monthly Contest 4
Due January 6, 2009

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 4
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Each square of a 100×100 grid is colored black or white so that there is at least one square of each color. Prove that there is a point with is a vertex of exactly one black square.

2. Let a and b be nonzero real numbers. Prove that at least one of the following inequalities is true:
$$\left|\frac{a+\sqrt{a^2+2b^2}}{2b}\right| < 1 \tag{1}$$
$$\left|\frac{a-\sqrt{a^2+2b^2}}{2b}\right| < 1 \tag{2}$$

3. In acute triangle ABC, the three altitudes meet at H. Given that $AH = BC$, calculate at least one of the angles of $\triangle ABC$.

4. Let x be an integer greater than 2. Prove that the binary representation of $x^2 - 1$ has at least three consecutive identical digits (000 or 111).

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f\left(x(1+y)\right) = f(x)\left(1+f(y)\right)$$
for all $x,\ y \in \mathbb{R}$.