Berkeley Math Circle
Monthly Contest 3
Due December 2, 2008

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 3
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Find all positive integers p such that p, $p+4$, and $p+8$ are all prime.

2. Each vertex of a regular heptagon is colored either red or blue. Prove that there is an isosceles triangle with all its vertices the same color.

3. Let a, b, and c be positive real numbers satisfying $a^b > b^a$ and $b^c > c^b$. Does it follow that $a^c > c^a$?

4. Let n be a positive integer and let S be the set $1,\, 2, \ldots,\, n$. Define a function $f : S \to S$ by

$$f(x) = \begin{cases} 2x & \text{if } 2x \leq n, \\ 2n - 2x + 1 & \text{otherwise.} \end{cases}$$

   Define $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, and so on. If m is a positive integer satisfying $f^m(1) = 1$, prove that $f^m(k) = k$ for all $k \in S$.

5. *This problem was invalid on the contest. Correct formulation as of December 9.* Let ω_1, ω_2, and ω_3 be three circles passing through the origin O of the coordinate plane but not tangent to each other or to either axis. Denote by $(x_i, 0)$ and $(0, y_i)$, $1 \leq i \leq 3$, the respective intersections (besides O) of circle ω_i with the x and y axes. Prove that ω_1, ω_2, and ω_3 have a common point $P \neq O$ if and only if the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear.