Berkeley Math Circle
Monthly Contest 2 – Solutions

1. A building has seven rooms numbered 1 through 7, all on one floor, and any number of doors connecting these rooms. These doors may be one-way, admitting motion in only one of the two directions, or two-way. In addition, there is a two-way door between room 1 and the outside, and a treasure in room 7. Your object is to choose the arrangement of the rooms and the locations of the doors in such a way that

   (a) it is possible to enter room 1, reach the treasure, and make it back outside,

   (b) the minimum number of steps required to to this (each step consisting of walking through a door) is as large as possible.

   Solution. On the way to the treasure, no room need be entered twice; otherwise the path could be shortened by skipping the loop. Thus, the minimal path to the treasure, if it exists, is at most 7 steps long. Similarly, the minimal path from the treasure to the outside is at most 7 steps long, so the total number of steps cannot exceed 14. The arrangement of the rooms in a line, with 1 and 7 at opposite ends and two-way doors between all adjacent rooms, shows that 14 is attainable.

2. Prove that there is exactly one way to place circles in four of the blank squares of the cross-equation puzzle at right such that, no matter what natural numbers are placed in the circled squares, the five uncircled squares can be filled with natural numbers that make the three horizontal and three vertical equations true.

a	+	b	=	c
+		=		+
d	=	e	+	f
=		+		=
g	+	h	=	k

   Solution. If a, e, f, and h are circled, the equations

$$b = e + h,\ d = e + f,\ g = d + a,\ c = b + a$$

   define b, d, g, and c in such a way that the first two horizontal and first two vertical equations are satisfied. Then since

$$g + h = d + a + h = e + f + a + h = b + f + a = c + f,$$

   a value of k can be found which completes the puzzle. Note that b, c, d, g, and k are the sum of two other numbers in the puzzle. If any of them is circled and then filled with a 1, the puzzle will be unsolvable since no two natural numbers add to 1. Thus a, e, f, and h is the only permissible circled quadruplet.

3. A number is called a *j-half* if it leaves a remainder of j when divided by $2j + 1$.

   (a) Prove that for any k, there is a number which is simultaneously a j-half for $j = 1, 2, \ldots, k$.

   (b) Prove that there is no number which is a j-half for all positive integers j.

   Solution.

   (a) Note that a number n is a j-half if and only if $2n$ has a remainder of $2j$ when divided by $2j + 1$, which happens exactly when $2n + 1$ is divisible by $2j + 1$. Thus

$$n = \frac{3 \cdot 5 \cdot 7 \cdots (2k + 1) - 1}{2},$$

   which is clearly an integer, is a j-half for $j = 1, 2, 2, \ldots, k$.

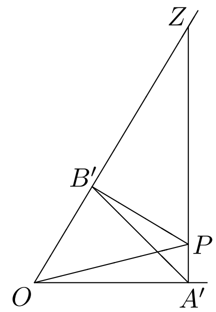
   (b) If n is any positive integer, take $j > n$. Then n obviously leaves a remainder of n when divided by $2n + 1$, not j as was desired.

4. Let AOB be a 60-degree angle. For any point P in the interior of $\angle AOB$, let A' and B' be the feet of the perpendiculars from P to AO and BO respectively. Denote by r and s the distances OP and $A'B'$. Find all possible pairs of real numbers (r, s).

Solution. Extend $A'P$ to meet OB at Z. Notice that, because $\angle OA'P$ and $\angle OB'P$ are both right, the circle with diameter OP passes through O, P, A', and B'. Thus $\angle B'OP = \angle B'A'P$ since both intercept the same arc on this circle, and $\triangle ZOP \sim \triangle ZA'B'$ by AA. We get

$$\frac{s}{r} = \frac{B'A'}{OP} = \frac{ZA'}{ZO} = \frac{\sqrt{3}}{2}$$

because ZOA' is a 30-60-90 triangle. Since r can obviously take on any value, the possibilities for (r, s) are $(r, \frac{r\sqrt{3}}{2})$ for every positive real r.



5. Prove that for every positive integer n, there is an integer x such that $x^2 - 17$ is divisible by 2^n.

Solution. We prove this by induction on n. If $n = 1$, 2, or 3, then $x = 1$ works. Suppose that $x^2 - 17$ is divisible by 2^n and $n \geq 3$. We seek to find y such that $y^2 - 17$ is divisible by 2^{n+1}. Let

$$x^2 - 17 = k \cdot 2^n.$$

If k is even, we are done since $x^2 - 17$ is divisible by 2^{n+1}. If k is odd, $k = 2m + 1$, we have

$$\begin{aligned}
\left(x + 2^{n-1}\right)^2 - 17 &= x^2 + 2 \cdot x \cdot 2^{n-1} + 2^{2(n-1)} - 17 \\
&= x^2 - 17 + x \cdot 2^n + 2^{2n-2} \\
&= (2m + 1) \cdot 2^n + x \cdot 2^n + 2^{2n-2} \\
&= 2^n (x + 1) + 2^{n+1} \left(m + 2^{n-3}\right).
\end{aligned}$$

Since x is obviously odd and $n \geq 3$, this is a multiple of 2^{n+1}, and $y = x + 2^{n-1}$ works.