# Monovariants in Action. Part II

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Some History. Just like in the previous Monovariant Part I session, most of the problems in this session were originally assembled and presented by Gabriel Carroll at the Berkeley Math Circle (BMC) on October 15, 2000. A session by Gabriel Carroll based on these notes will appear in volume II of *A Decade of the Berkeley Math Circle: the American Experience*, edited by Zvezdelina Stankova and Tom Rike, to be published by the American Mathematical Society. There are many contributors to the book, including most of the BMC instructors.

## 1. Numerical Monovariants

**Problem 1** (Russia'61)**.** A rectangular $m \times n$ array of real numbers is given. Whenever the sum of the numbers in any row or column is negative, we may switch the signs of all the numbers in that row or column, from negative to positive or vice versa. Prove that if we repeat this operation, eventually all the row and column sums will be nonnegative.

**Problem 2** (St. Petersburg'93)**.** Several positive integers are written on a blackboard. One can erase any two distinct numbers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

**Problem 3** (Cofman)**.** Place four nonnegative integers $a, b, c, d$ around a circle. For any two consecutive numbers, take their absolute difference, and write that difference between them; then erase the four original numbers. Thus, after one step, the four new numbers will be $|a-b|, |b-c|, |c-d|, |d-a|$. Iterate this process. Is it true that the process always eventually leads to a circle with four 0s? (After solving this, you might try to generalize your result by replacing 4 with any power of 2 greater than one.)

**Problem 4** (IMO'86)**.** An integer is written at each vertex of a regular pentagon so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers $x, y, z$ with $y < 0$, then the following operation is allowed: the numbers $x, y, z$ are replaced by $x+y, -y, z+y$, respectively. Such an operation is repeated as long as at least one of the five numbers is negative. Determine whether the procedure necessarily comes to an end in a finite number of steps.

**Problem 5** (USAMO'93)**.** Let $a$ and $b$ be two odd positive integers. Define a sequence by putting $f_1 = a$, $f_2 = b$, and letting $f_{n+1}$ for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Prove that $f_n$ becomes constant for $n$ sufficiently large, and determine the eventual value as a function of $a$ and $b$.

**Problem 6** (USAMO'97)**.** Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let $x_0$ be a real number between 0 and 1. For each positive integer $k$, define $x_k = 0$ if $x_{k-1} = 0$, and $x_k = \{p_k/x_{k-1}\}$ otherwise, where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of $x$. Find, with proof, all $x_0$ satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

**Problem 7** (USAMO'07)**.** Let $n$ be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting $a_k$ be the unique integer in the range $0 \leq a_k \leq k-1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by $k$. For instance, when $n = 9$ the sequence obtained is $9, 1, 2, 0, 3, 3, 3, \ldots$. Prove that for any $n$ the sequence $a_1, a_2, a_3, \ldots$ eventually becomes constant.

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