Full Stop due to Monovariants

by Zvezdelina Stankova[1]

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Some History. Most of the problems in this session were originally assembled and presented by Gabriel Carroll at the Berkeley Math Circle (BMC) on October 15, 2000. Gabriel was a math circler as well as an instructor and monthly contest coordinator at BMC. He received the grand top prizes at three consecutive Bay Area Math Olympiads (BAMO), and participated on the USA team at the International Math Olympiads, winning two gold and one silver medals, including one of only 4 perfect scores at the IMO'2001 in Washington, DC, and was a top Putnam winner four times. He has also created a number of BAMO problems, including the last one on this handout. Among his other talents, he is a virtuoso piano player, loves to draw cartoons and edited the Harvard-based humor magazine Swift, taught English in Hunan province in China, worked for a research company in Massachusetts, and will be starting a Ph.D. program in Economics at MIT in fall 2007.

A session by Gabriel Carroll based on these notes will appear in volume I of

A Decade of the Berkeley Math Circle:
the American Experience

edited by Zvezdelina Stankova and Tom Rike, to be published by the American Mathematical Society. There are many contributors to the book, including most of the BMC instructors.

1. Who Wants to Eat Most?

Problem 1 (Chocolate)**.** Two players take turns breaking up a 3×4 chocolate bar. At each turn, a player picks up a piece and breaks it along a division between its squares. Eventually, the bar will split into 12 squares and the game will end. The player who makes the last break wins and eats the chocolate. Is there a strategy for one of the two players to win?

2. The Realm of Mansion Problems

Problem 2 (Generic Mansion)**.** 2000 people reside in the rooms of a 123-room mansion. Each minute, as long as not all the people are in the same room, somebody walks from one room into a different room with at least as many people in it. Prove that eventually all the people will gather in one room.

Problem 3 (Random Mansion)**.** 2000 people are distributed among the 123 rooms of a mansion. Each minute, as long as the people are not all in one room, one or more people from the same room leave that room. At least one of these people goes to a room with at least as many people as the original room. (The rest of the people in the original room stay, or move to any other room in the mansion.) Prove that eventually all the people will be in the same room.

Problem 4 (Gender Mansion)**.** 1000 men and 1000 women are distributed among the rooms of a 123-room mansion. They move among the rooms according to the following rules: either

- a man moves from a room with more men than women into a room with more women than men (counted before he moves); or
- a woman moves from a room with more women than men into a room with more men than women.

Prove that eventually the people will stop moving.

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Problem 5 (Hybrid Mansion). There are 1000 women and 1000 men in a 123-room mansion, as usual. This time, they walk among the rooms according to the following rules: at each step, either

- a man moves from a room with more men than women into a room with more women than men (counted before he moves); or
- a woman moves from a room with more women than men into another room with at least as many total people.

Prove that eventually the people will stop moving.

Problem 6 (Refined Mansion). What is the *smallest number of steps* after which the room-migration process in original mansion Problem 2 is guaranteed to have terminated?

3. The Realm of Frog Problems

Problem 7 (Linear Frogs). Some number of frogs are squatting on a row of 2000 lily pads in a swamp. Each minute, if there are two frogs on the same lily pad, and this pad is not at either end of the row, the two frogs may jump to the two adjacent lily pads (in opposite directions). Prove that this process cannot be repeated forever.

Problem 8. (Circle of Frogs[2]). A circular swamp is divided into 2000 sectors. There are 2001 frogs in the swamp. Each minute, some two frogs that are in the same sector jump (in opposite directions) to the two adjacent sectors. Prove that at some point there will be at least 1001 different sectors containing frogs.

4. Gosh! How could Gabriel Come up with Such Problems?

Problem 9 (BAMO 2006, Carroll). We have k switches arranged in a row, and each switch points up, down, left, or right. Whenever three successive switches all point in different directions, all three may be simultaneously turned so as to point in the fourth direction. Prove that this operation cannot be repeated infinitely many times.

5. Which Functions are Suitable for Monovariants and When?

Problem 10. (Smiles vs. Frowns).

(1) "Smily" (concave up) functions yield increasing monovariants when the data spreads out, and decreasing monovariants when the data clusters together.
(2) "Frowny" (concave down) functions yield decreasing monovariants when the data spreads out, and increasing monovariants when the data clusters together.

6. Something for the Die-Hards

Problem 11 (Conway Checkers). Imagine that you have an infinite square grid, with a particular horizontal line of the grid designated. You play the following game:

- First, you may initially place checkers in the squares below the line — as many as you want, but no more than one checker per square.
- Then, you may take a checker and jump it over a checker that is adjacent to it — in any of the four directions — into the square immediately beyond, if that space is vacant. In the process, you remove the checker that has been jumped over. You may continue jumping checkers, as long as there are two checkers adjacent to each other somewhere.

The goal is to get some checker to be as far above the designated line as possible. What is the highest row that can be reached?

[2] adapted from Jiangsu (China) Math Olympiad 1993