ELEMENTARY GEOMETRY PROBLEMS ON LOCI

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1. Basic Facts and Introductory Problems

Theorem 1.1. *The locus of points equidistant from two points A and B is a line perpendicular to the segment AB and passing through its midpoint.*

Theorem 1.2. *The locus of points at distance R from point O is a circle with radius R and center O.*

Theorem 1.3. *The locus of points from which a given segment AB is seen in a given angle is the union of two arch of circles symmetric with respect to AB.*

Problem 1.4. a) Find the locus of points equidistant from two parallel lines?
a) Find the locus of points equidistant from two crossing lines?

Problem 1.5. Find the locus of midpoints of segments with endpoints on two parallel lines.

Problem 1.6. Given triangle $\triangle ABC$, find the locus of points X such that $AX \leq BX \leq CX$.

Problem 1.7. Find the locus of points X such that the tangents from X to a given circle have a given length.

Problem 1.8. A point A on a circle is fixed. Find the locus of points X that divide chords with A as an endpoint in the ratio of $1:2$ counting from A.

2. Problems

Problem 2.1. Given two lines that meet at point O. Find the locus of points X for which the sum of the lengths of projections of segment OX to these lines is a constant.

Problem 2.2. Sides AB and CD of quadrilateral $ABCD$ of area S are not parallel. Find the locus of points X inside the quadrilateral for which $S_{ABX} + S_{CDX} = \frac{1}{2}S$.

Problem 2.3. Given points A and B in the plane, find the locus of points M for which the difference of the squared lengths of segments AM and BM is a constant.

Problem 2.4. Given two non-intersecting circles, find the locus of the centers of circles that divide the given circles in halves (i.e., that intersect the given circles in diametrically opposite points).

[1]Most of the problems are taken from an excelent plane geometry book by V. Prasolov, whose free English translation can be found at http://students.imsa.edu/ tliu/Math/planegeo.pdf

Problem 2.5. A segment moves along the plane so that its endpoints slide along the legs of a right angle $\angle ABC$. What is the trajectory traversed by the midpoint of this segment? (We naturally assume that the length of the segment does not vary while it moves.)

Problem 2.6. Two points, A and B in plane are given. Find the locus of points M for which $|AM| : |BM| = k$.

Problem 2.7. Points A and B on a circle are fixed. Point C runs along the circle. Find the set of the intersection points of the medians of triangles $\triangle ABC$.

Problem 2.8. Triangle $\triangle ABC$ is given. Find the locus of the centers of rectangles $PQRS$ whose vertices Q and P lie on side AC and vertices R and S lie on sides AB and BC, respectively.

Problem 2.9. Points P and Q move with the same constant speed v along two lines that intersect at point O. Prove that there exists a fixed point A in plane such that the distances from A to P and Q are equal at all times.

Problem 2.10. Inside a convex polygon points P and Q are taken. Prove that there exists a vertex of the polygon whose distance from Q is smaller than that from P.

Problem 2.11. In plane, two non-intersecting disks are given. Does there necessarily exist a point M outside these disks that satisfies the following condition: each line that passes through M intersects at least one of these disks? Find the locus of points M with this property.

Problem 2.12. Points M and N are such that $AM : BM : CM = AN : BN : CN$. Prove that line MN passes through the center O of the circumscribed circle of triangle $\triangle ABC$.

Problem 2.13. Given two circles with centers O_1 and O_2 and a line l intersects these two circles at four points; A_1, B_1 (belong to the first circle) and A_2, B_2 (belong to the second circle). In addition, no two tangent lines at these points are parallel. Consider the four intersection points between tangents to different circles at these points. Show that they lie on the same circle whose center lies on the line O_1O_2.

Problem 2.14. Given a convex quadrilateral $ABCD$, what is the locus of points M such that $S_{ABMD} = S_{CBMD}$.

Problem 2.15. Through the midpoint of each diagonal of a convex quadrilateral a line is drawn parallel to the other diagonal. These lines meet at point O. Prove that segments that connect O with the midpoints of the sides of the quadrilateral divide the area of the quadrilateral into four equal parts.

3. Extra (Harder) Problems

Problem 3.1. Four points in the plane are given. Find the locus of the centers of rectangles formed by four lines that pass through the given points.

Problem 3.2. a) Points A and B on a circle are fixed and points A_1 and B_1 run along the same circle so that the value of arc A_1B_1 remains a constant; let M be the intersection point of lines AA_1 and BB_1. Find the locus of points M.

b) Triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are inscribed in a circle; triangle $\triangle ABC$ is fixed and triangle $A_1B_1C_1$ rotates. Prove that lines AA_1, BB_1 and CC_1 intersect at one point for not more than one position of triangle $\triangle A_1B_1C_1$.

Problem 3.3. Prove that the perpendiculars dropped from points A_1, B_1 and C_1 to sides BC, CA, AB of triangle ABC intersect at one point if and only if

$$A_1B^2 + C_1A^2 + B_1C^2 = B_1A^2 + A_1C^2 + C_1B^2.$$

Problem 3.4. Triangle ABC is an equilateral one, P an arbitrary point. Prove that the perpendiculars dropped from the centers of the inscribed circles of triangles PAB, PBC and PCA to lines AB, BC and CA, respectively, meet at one point.

Problem 3.5. Prove that if perpendiculars raised at the bases of bisectors of a triangle meet at one point, then the triangle is an isosceles one.

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