Berkeley Math Circle — Oct. 2, 2007

Survival Combinatorics

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Beginner Problems

1 An n-bit string is n-digit "word," each digit of which is either 0 or 1. How many n-bit strings contain at least 1 zero?

2 Imagine a piece of graph paper. Starting at the origin draw a path to the point $(10,10)$, that stays on the grid lines (which are one unit apart) and has a total length of 20. For example, one path is to go from $(0,0)$ to $(0,7)$ to $(4,7)$ to $(4,10)$ to $(10,10)$. Another path goes from $(0,0)$ to $(10,0)$ to $(10,10)$. How many possible different paths are there?

3 In how many ways can two squares be selected from an 8-by-8 chessboard so that they are not in the same row or the same column?

4 Suppose we have three different toys and we want to give them away to two girls and one boy (one toy per child). The children will be selected from 4 boys and 6 girls. In how many ways can this be done?

5 Suppose again that we have three different toys and we want to give them away (one toy per kid) to three children selected from a pool of 4 boys and 6 girls, but now we require that at least two boys get a toy. In how many ways can this be done?

6 How many different ordered triples (a,b,c) of non-negative integers are there such that $a+b+c=50$? What if the three integers had to be positive?

7 How many ways can the positive integer n can be written as an ordered sum of at least one positive integer? For example,

$$4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1,$$

so when $n=4$, there are 8 such *ordered partitions*.

8 Ten children order ice cream cones at a store featuring 31 flavors. How many orders are possible in which at least two children get the same flavor?

9 In how many ways can four squares, not all in the same row or column, be selected from an 8-by-8 chessboard to form a rectangle?

10 In how many ways can we place r red balls and w white balls in n boxes so that each box contains at least one ball of each color?

11 Given n points arranged around a circle and the chords connecting each pair of points is drawn. If no three chords meet in a point, how many points of intersection are there? For example, when $n = 6$, there are 15 intersections.

12 Eight people are in a room. One or more of them get an ice-cream cone. One or more of them get a chocolate-chip cookie. In how many different ways can this happen, given that at least one person gets both an ice-cream cone and a chocolate-chip cookie?

13 How many *strictly increasing* sequences of positive integers begin with 1 and end with 1000?

14 How many subsets of the set $\{1, 2, 3, 4, \ldots, 30\}$ have the property that the sum of the elements of the subset is greater than 232?

15 Prove that for all positive integers n,

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} = n2^{n-1}.$$

16 Ten dogs encounter 8 biscuits. Dogs do not share biscuits! How many different ways can the biscuits can be consumed

(a) if we assume that the dogs are distinguishable, but the biscuits are not;

(b) if we assume that the dogs and the biscuits are distinguishable (for example, each biscuit is a different flavor).

(c) if we assume that neither the dogs nor the biscuits are distinguishable? (We are able to distinguish dogs from biscuits, however. The answer is *not* 1!)

17 Decide whether there exist 10,000 ten-digit numbers divisible by seven, all of which can be obtained from one another by a reordering of their digits.

18 Sal the Magician asks you to pick any five cards from a standard deck. You do so, and then show them to Sal's assistant Pat, who places one of the five cards back in the deck and then puts the remaining four cards into a pile. Sal is blindfolded, and does not witness any of this. Then Sal takes off the blindfold, takes the pile of 4 cards, reads the four cards that Pat has arranged, and is able to find the fifth card in the deck (even if you shuffle the deck after Pat puts the card in the deck). Assume that neither Sal nor Pat have supernatural powers, and that the deck of cards is not marked, and that the cards are all in neat pile, all facing the same way. How is the trick done? Harder version: you pick which of the five cards goes back into the deck (instead of Pat)

19 Let T be the set of all positive integers whose base-10 representation does not contain any zeros. Prove or disprove: The sum of the reciprocals of the members of T diverges.

Harder Problems

1 There are 10 adjacent parking spaces in the parking lot. When you arrive in your new Rolls Royce, there are already seven cars in the lot. What is the probability that you can find two adjacent unocupied spaces for your Rolls? Generalize.

2 Define a **domino** to be a 1×2 rectangle. In how many ways can a $n \times 2$ rectangle be tiled by dominos?

3 Find the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive elements of $\{1, 2, \ldots, n\}$.

4 How many five-card hands from a standard deck of cards contain at least one card in each suit?

5 Four young couples are sitting in a row. In how many ways can we seat them so that no person sits next to their "significant other?"

6 Let S be a set with n elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a, c\}, \{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}, \{a, c\}$.

7 (USAMO 72) A random number generator randomly generates the integers $1, 2, \ldots, 9$ with equal probability. Find the probability that after n numbers are generated, the product is a multiple of 10.

8 Let $a_1, a_2, \ldots, a_n$ be an ordered sequence of n distinct objects. A **derangement** of this sequence is a permutation that leaves no object in its original place. For example, if the original sequence is $1, 2, 3, 4$, then $2, 4, 3, 1$ is not a derangement, but $2, 1, 4, 3$ is. Let D_n denote the number of an n-element sequence. Show that

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right).$$

9 How many nonnegative integer solutions are there to $a + b + c + d = 17$, provided that $d \leq 12$?

10 Use a combinatorial argument to show that for all positive integers n, m, k with n and m greater than or equal to k,

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{n+m}{k}.$$

This is known as the **Vandermonde Convolution Formula**.

11 (Putnam 1996) Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1,2,\ldots,n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

12 Imagine that you are going to give n kids ice cream cones, one cone per kid, and there are k different flavors available. Assuming that no flavors get mixed, show that the number of ways we can give out the cones *using all k flavors* is

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \cdots + (-1)^k\binom{k}{k}0^n.$$

13 (IMO 89) Let a permutatation π of $\{1,2,\ldots,2n\}$ have property P if

$$|\pi(i) - \pi(i+1)| = n$$

for at least one $i \in [2n-1]$. Show that, for each n, there are more permutations with property P than without it.

14 (Putnam 1996) Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

15 Define $p(n)$ to be the number of different ways a positive integer n can be written as a sum of positive integers, where the order of the summands doesn't matter. Here is a table of the first few values of $p(n)$. Show that $p(n) \geq 2^{\lfloor\sqrt{n}\rfloor}$ for all $n \geq 2$.

16 (Putnam 1993) Let P_n be the set of subsets of $\{1,2,\ldots,n\}$. Let $c(n,m)$ be the number of functions $f : P_n \to \{1,2,\ldots,m\}$ such that $f(A\cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n,m) = \sum_{j=1}^{m} j^n.$$

17 For each positive integer n, let a_n be the number of permutations τ of $\{1,2,\ldots,n\}$ such that $\tau(\tau(\tau(x))) = x$ for $x = 1,2,\ldots,n$. The first few values are

$$a_1 = 1, a_2 = 1, a_3 = 3, a_4 = 9.$$

(a) Prove that 3^{334} divides a_{2001}.

(b) Improve on the above, by finding the best constant c such that, for all (or for all large enough) n, 3^{cn} divides a_n.

(c) Improve on the above by finding a bijective solution.