berkeley math circle

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combinatorial geometry
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1. Given five points in the plane, no three of which are collinear, prove that some four of them form a convex quadrilateral.
2. Given any ten points in a square of side 1, prove that you can find two of them whose distance is at most $\sqrt{2}/3$.
3. Given any thirteen points in an equilateral triangle of area 1, prove that you can find three of them such that the triangle they form has area at most $\frac{1}{4}$.
4. If each point in the plane is colored red or blue, prove that there exists a rectangle whose four vertices have the same color.

Arrangements of lines.

5. Suppose we draw some lines in the plane. Prove that the regions that are formed can be colored black and white so that any two regions which have a common segment are assigned different colors.
6. If we draw n lines in the plane, what is the smallest number of regions that we can form? What is the largest number of regions that we can form?
7. If we make n straight cuts to an apple, what is the smallest number of regions that we can form? What is the largest number of regions that we can form? What is the largest number of regions that we can form which do not contain part of the skin of the apple?
8. If we choose n points on the circumference of a circle and draw all the $\binom{n}{2}$ chords formed by them, what is the largest number of regions that we can form inside the circle?
9. Given are 8 distinct points in the plane. Construct all 28 possible segments with the 8 points as endpoints, and their perpendicular bisectors. It is known that at least 22 perpendicular bisectors of these segments concur. Show that all of them concur.

Halving lines and circles.

10. Given are $2n$ points in the plane, no three of which are on a line. Prove that there is at least one *halving line*: a line going through two of the points which has $n-1$ of the points on one side and $n-1$ of the points on the other side.
11. Given are $2n$ points in the plane, no three of which are on a line. Prove that there are at least n halving lines.
12. Given are $2n+1$ points in the plane, no three of which are on a line and no four of which are on a circle. Prove that there is at least one *halving circle*: a circle going through three of the points which has $n-1$ of the points inside the circle and $n-1$ of the points outside the circle.

13. Given are $2n+1$ points in the plane, no three of which are on a line and no four of which are on a circle. Prove that for any two points A and B there is at least one halving circle that goes through them. Conclude that there are at least $\binom{2n+1}{2}/3$ halving circles.

Some harder ones

14. (IMO 02) Consider the "staircase" of all points (x, y) with x, y non-negative integers such that $x + y < n$. Each element of S is colored red or blue, so that if (x, y) is red then any point (x', y') weakly southwest of it ($x' \leq x$ and $y' \leq y$) is also red.

    An X-set is a collection of n blue elements on different rows of the staircase. A Y-set is a collection of n blue elements on different columns of the staircase. Prove that the number of X-sets equals the number of Y-sets.

15. (Iberoamerican Math Olympiad 1997) Given are 1997 points inside a circle of radius 1, one of which is the center of the circle. For each point take the distance to the closest (distinct) point. Show that the sum of the squares of the resulting distances is at most 9.

16. (IMO 00 shortlist, Colombian Math Olympiad 01) Let $S = \{P_1, \ldots, P_n\}$ be a set of points in the plane, no three of which are on a line and no four of which are on a circle. For $1 \leq t \leq n$, let a_t be the number of circles $P_iP_jP_k$ which contain P_t in their interior.

    Prove that, knowing only the values of n and $a_1 + \cdots + a_n$, it is possible to determine whether the points of S are the vertices of a convex polygon.

Some harder ones on halving circles

17. (IMO 99 shortlist) Given are 5 points in the plane, no three of which are on a line and no four of which are on a circle. Prove that there are exactly 4 halving circles.

18. (APMO 99) Given are $2n+1$ points in the plane, no three of which are on a line and no four of which are on a circle. Prove that for any two points A and B there is an odd number of halving circles that go through them. Conclude that the number of halving circles has the same parity as n.

19. (See [1].) Given are $2n+1$ points in the plane, no three of which are on a line and no four of which are on a circle. Prove that the number of halving circles is exactly n^2.

[1] F. Ardila. *The number of halving circles.* American Mathematical Monthly, 111 (2004), 586-592. Available at `http://math.sfsu.edu/federico/circles.pdf`.