Berkeley Math Circle
Monthly Contest 7 – Solutions

1. Find all positive prime numbers p such that $p+2$ and $p+4$ are prime as well.

   Hint. Show that for most prime numbers p, either $p+2$ or $p+4$ is divisible by 3.

   Solution. For $p=3$, $p+2=5$, $p+4=7$ and these are obviously prime. For $p>3$, we know that p is not divisible by 3. The remainder of p when divided by 3 can be either 1 or 2. If it is one, then $p+2$ is divisible by 3, if it is 2, then $p+4$ is divisible by 3. Hence $p=3$ is the only solution.

2. Let P be the point inside the square $ABCD$ such that $\triangle ABP$ is equilateral. Calculate the angle $\angle CPD$. Explain your answer!

   Solution. The triangle DAP is isosceles because $AD = AP$ hence $\angle ADP = \angle APD = \frac{180^\circ - \angle DAP}{2} = 75^\circ$. Hence $\angle PDC = 15^\circ$. Similarly $\angle DCP = 15^\circ$ and hence $\angle CPD = 180^\circ - 2\cdot 15^\circ = 150^\circ$.

3. Find at least one non-zero polynomial $P(x,y,z)$ such that $P(a,b,c)=0$ for every three real numbers that satisfy $\sqrt[3]{a}+\sqrt[3]{b}=\sqrt[3]{c}$.

   Remark. Polynomial in three variables refers to any expression built from x, y, z and numerlas using only addition, subtraction, and multiplication. Parentheses or positive integer exponents, as in $x(y+z)^2$ are allowed since this can be expanded to $xyy+2xyz+xzz$.

   Solution. Cube both sides of the condition $\sqrt[3]{x}+\sqrt[3]{y}=\sqrt[3]{z}$:

$$x + 3\sqrt[3]{x}\sqrt[3]{x}\sqrt[3]{y} + 3\sqrt[3]{x}\sqrt[3]{y}\sqrt[3]{y} + y = z$$

$$3\sqrt[3]{x}\sqrt[3]{x}\sqrt[3]{y} + \sqrt[3]{x}\sqrt[3]{y}\sqrt[3]{y} = z - x - y$$

$$3\sqrt[3]{x}\sqrt[3]{y}(\sqrt[3]{x}+\sqrt[3]{y}) = z - x - y$$

   But since $\sqrt[3]{x}+\sqrt[3]{y}=\sqrt[3]{z}$,

$$3\sqrt[3]{x}\sqrt[3]{y}\sqrt[3]{z} = z - x - y$$

$$27xyz = (z-x-y)^3.$$

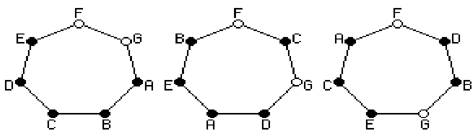
   Hence $P(x,y,z) = 27xyz - (z-x-y)^3$ is one such polynomial.

4. If $f(1)=1$ and $f(1)+f(2)+\cdots+f(n) = n^2 f(n)$ for every integer $n \geq 2$, evaluate $f(2008)$.

   Solution. $n^2 f(n) - f(n) = f(1)+f(2)+\cdots+f(n-1) = (n-1)^2 f(n-1)$ hence $f(n) = \frac{(n-1)^2}{n^2-1} f(n-1) = \frac{n-1}{n+1} f(n-1)$. Thus $f(2008) = \frac{2007}{2009} f(2007) = \frac{2007}{2009}\cdot\frac{2006}{2008} f(2006) = \frac{2007}{2009}\cdot\frac{2006}{2008}\cdot\frac{2005}{2007} f(2005) = \cdots = \frac{2007!}{2009\cdot 2008\cdot\cdots\cdot 4\cdot 3} f(1) = \frac{2}{2009\cdot 2008}$.

5. Given five vertices of a regular heptagon, construct the two remaining vertices using straightedge alone.

   Solution. Let A, B, C, D, and E be the known vertices and F and G the unknown vertices. The arrangement of A, B, C, D, and E depends on the relative positions of F and G as shown in the diagram. The following construction applies to all three cases.



   By connecting the intersections of BC with DE and BD with CE one finds l, the line of symmetry through G. Let AB and AC meet l in H and I respectively. By symmetry about l, HE and ID both pass through F. Now perform the same construction on B, C, D, E, F to get G.

   Remark. In general, given five consecutive vertices A, B, C, D, E of a regular polygon or regular star polygon with at least seven sides, the next F can be found by this construction, except that l will not necessarily pass through G, although it is always the perpendicular bisector of CD, BE and AF and is a line of symmetry of the polygon.