Berkeley Math Circle
Monthly Contest 6 – Solutions

1. Ten different points are marked on a circle. Two players A and B play the following game. A moves first and the players alternate their moves. In each of the moves a player connects two of the points with a straight line segment. A player whose segment crosses a segment previously drawn will lose the game. Which player has a winning strategy and what is the strategy.

   Solution. Notice that a player can draw a line segment if and only if the 10-gon is not partitioned into triangles. Since there is a total of 17 segments for any partition of 10-gon into triangles, the first player will win the game no matter how he plays.

2. Prove that no integer greater than 2008 can be equal to the sum of squares of its digits.

   Solution. Let $n = \overline{a_k a_{k-1} \dots a_0}$ be an integer equal to the sum of squares of its own digits. Then $n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + a_0$. On the other hand $a_0^2 + a_1^2 + \cdots + a_k^2 \leq 9^2 \cdot (k+1) = 81 \cdots (k+1) < 10^k$ for $k \geq 4$ (the last inequality is easy to prove by induction because it holds for $k = 4$ and if it holds for some k then $81 \cdot (k+2) = 81 \cdot (k+1) + 81 \leq 10^k + 81 < 10^k + 10^k < 10^{k+1}$).

3. If $x \geq 4$ is a real number prove that
$$\sqrt{x} - \sqrt{x-1} \geq \frac{1}{x}.$$

   Solution. Notice that
$$\sqrt{x} - \sqrt{x-1} = \frac{(\sqrt{x} - \sqrt{x-1}) \cdot (\sqrt{x} + \sqrt{x-1})}{\sqrt{x} + \sqrt{x-1}} = \frac{1}{\sqrt{x} + \sqrt{x-1}}.$$
   Now the required inequality is equivalent to $x \geq \sqrt{x} + \sqrt{x-1}$ or after dividing both sides by $\sqrt{x}$:
$$\sqrt{x} \geq 1 + \sqrt{1 - \frac{1}{x}}.$$
   The last inequality holds for $x \geq 4$ because $1 + \sqrt{1 - \frac{1}{x}} \leq 2 \leq \sqrt{x}$.

4. Wally has a very unusual combination lock number. It has five digits, all different, and is divisible by 111. If he removes the middle digit and replaces it at the end, the result is a larger number that is still divisible by 111. If he removes the digit that is now in the middle and replaces it at the end, the result is a still larger number that is still divisible by 111. What is Wally's combination lock number? Explain your answer!

   Solution. The solution is 74259. The numbers 74259, 74592, and 74925 are all divisible by 111. Denote the original number by $\overline{abcde}$ (the line prevents confusion with $a \cdot b \cdot c \cdot d \cdot e$). Then we have
$$111 | \overline{abcde}$$
$$111 | \overline{abdec}$$
   Subtracting,
$$111 | \overline{abdec} - \overline{abcde}$$
$$111 | \overline{dec} - \overline{cde}$$
   Since the number $\overline{dec} - \overline{cde} = 90d + 9e - 99c$ is divisible by 9 we get $33 | \overline{dec} - \overline{cde}$. $333 | \overline{dec} - \overline{cde}$. Since it is given that $\overline{abdec} > \overline{abcde}$, $\overline{cde} \leq \overline{dec} - 333$.

   However, we could have done the above logic with d, e, c instead of c, d, e and gotten $\overline{dec} \leq \overline{ecd} - 333$. Consequently $\overline{cde} \leq \overline{ecd} - 666$. Since obviously $\overline{ecd} \leq 999$, $\overline{cde}$ is one of the multiples of 37 up to 333:

   000 037 074 111 148 185 222 259 296 333

   We can immediately eliminate 000, 111, 222, and 333, since we know that the digits are all different. We can also eliminate 074, 185, and 296, since $\overline{ecd} > \overline{dec}$. The three remaining choices are all of the form $111 \cdot k + 37$. So
$$\begin{aligned} 11 | \overline{abcde} & \\ &= \overline{ab000} + \overline{cde} \\ &= 999 \cdot \overline{ab} + \overline{ab} + 111 \cdot k + 37 \\ 11 | \overline{ab} + 37 & \end{aligned}$$
   whence $\overline{ab} = 74$. This leaves only three possibilities for Wally's combination lock number: 74037, 74148, and 74259, of which only the last has all unlike digits.

5. Let $A_0, A_1, \dots, A_n$ be points in a plane such that

(i) $A_0A_1 \le \frac{1}{2}A_1A_2 \le \cdots \le \frac{1}{2^{n-1}}A_{n-1}A_n$ and

(ii) $0 < \measuredangle A_0A_1A_2 < \measuredangle A_1A_2A_3 < \cdots < \measuredangle A_{n-2}A_{n-1}A_n < 180^\circ$,

where all these angles have the same orientation. Prove that the segments A_kA_{k+1}, A_mA_{m+1} do not intersect for each k and n such that $0 \le k \le m-2 < n-2$.

Solution. Suppose that $A_kA_{k+1} \cap A_mA_{m+1} \neq \emptyset$ for some k, $m > k+1$. Without loss of generality we may suppose that $k = 0$, $m = n-1$ and that no two segments A_kA_{k+1} and A_mA_{m+1} intersect for $0 \le k < m-1 < n-1$ except for $k = 0$, $m = n-1$. Also, shortening A_0A_1, we may suppose that $A_0 \in A_{n-1}A_n$. Finally, we may reduce the problem to the case that $A_0 \dots A_{n-1}$ is convex: Otherwise, the segment $A_{n-1}A_n$ can be prolonged so that it intersects some A_kA_{k+1}, $0 < k < n-2$.

If $n = 3$, then $A_1A_2 \ge 2A_0A_1$ implies $A_0A_2 > A_0A_1$, hence $\angle A_0A_1A_2 > \angle A_1A_2A_3$, a contradiction.

Let $n = 4$. From $A_3A_2 > A_1A_2$ we conclude that $\angle A_3A_1A_2 > \angle A_1A_3A_2$. Using the inequality $\angle A_0A_3A_2 > \angle A_0A_1A_2$ we obtain that $\angle A_0A_3A_1 > \angle A_0A_1A_3$ implying $A_0A_1 > A_0A_3$. Now we have $A_2A_3 < A_3A_0 + A_0A_1 + A_1A_2 < 2A_0A_1 + A_1A_2 \le 2A_1A_2 \le A_2A_3$, which is not possible.

Now suppose $n \ge 5$. If α_i is the exterior angle at A_i, then $\alpha_1 > \cdots > \alpha_{n-1}$; hence $\alpha_{n-1} < \frac{360^\circ}{n-1} \le 90^\circ$. Consequently $\angle A_{n-2}A_{n-1}A_0 \ge 90^\circ$ and $A_0A_{n-2} > A_{n-1}A_{n-2}$. On the other hand, $A_0A_{n-2} < A_0A_1 + A_1A_2 + \cdots + A_{n-3}A_{n-2} < \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2}\right) A_{n-1}A_{n-2} < A_{n-1}A_{n-2}$, which contradicts the previous relation.