Berkeley Math Circle
Monthly Contest 5 – Solutions

1. The rectangle $MNPQ$ is inside the rectangle $ABCD$. The portion of the rectangle $ABCD$ outside of $MNPQ$ is colored in green. Using just a straightedge construct a line that divides the green figure in two parts of equal areas.

   Solution. The line passing through the centers of $ABCD$ and $MNPQ$ divides each of the rectangles in two pieces of equal areas. Hence that line will divide the grean part into equal areas. The line can be easily constructed using only a straightedge because we can construct the centers of the given rectangles as intersections of their diagonals.

2. Determine the positive real numbers a and b satisfying $9a^2 + 16b^2 = 25$ such that $a \cdot b$ is maximal. What is the maximum of $a \cdot b$? Explain your answer!

   Hint. If x and y are any two real numbers then $x^2 + y^2 \geq 2xy$.

   Solution. Applying the inequality $x^2 + y^2 \geq 2xy$ on $x = 3a$ and $y = 4b$ we get $25 = (3a)^2 + (4b)^2 \geq 2 \cdot 3a \cdot 4b = 24ab$. Hence $ab \leq \frac{25}{24}$. The equality is attained for $x = y$ or equivalently for $3a = 4b$. In that case $25 = (3a)^2 + (4b)^2 = 2 \cdot 9a^2$ hence $a = \frac{5}{3\sqrt{2}}$. Now we have $b = \frac{3}{4}a = \frac{5}{4\sqrt{2}}$.

3. Find all pairs of integers (x, y) for which $x^2 + xy = y^2$.

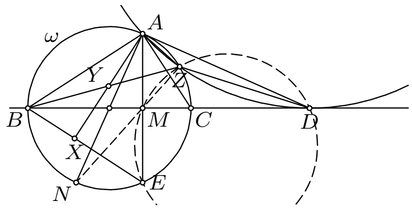
   Solution. The only such pair is (0,0). If $x = 0$, we easily get $y = 0$ which satisfies the equation. Otherwise dividing through by x^2 we get
$$1 + \frac{y}{x} = \left(\frac{y}{x}\right)^2.$$
   This implies $\left(\frac{y}{x}\right)^2 - \frac{y}{x} - 1 = 0$ and the quadratic formula gives us $\frac{y}{x} = \frac{1 \pm \sqrt{5}}{2}$. Hence $\frac{2y}{x} = 1 \pm \sqrt{5}$ or equivalently $\frac{2y - x}{x} = \pm\sqrt{5}$. Since $\sqrt{5}$ is irrational, this is a contradiction.

4. Let n be a positive integer. Prove that there exist distinct positive integers x, y, z such that
$$x^{n-1} + y^n = z^{n+1}.$$

   Solution. One solution is
$$x = 2^{n^2} 3^{n+1}, \qquad y = 2^{n^2 - n} 3^n, \qquad z = 2^{n^2 - 2n + 2} 3^{n-1}.$$

5. Let ABC be a triangle such that $\angle A = 90°$ and $\angle B < \angle C$. The tangent at A to its circumcircle ω meets the line BC at D. Let E be the reflection of A across BC, X the foot of the perpendicular from A to BE, and Y the midpoint of AX. Let the line BY meet ω again at Z. Prove that the line BD is tangent to the circumcircle of triangle ADZ.

   Solution. Let M be the point of intersection of AE and BC, and let N be the point on ω diametrically opposite A. Since $\angle B < \angle C$, points N and B are on the same side of AE.

   Furthermore, $\angle NAE = \angle BAX = 90° - \angle ABE$; hence the triangles NAE and BAX are similar. Consequently, $\triangle BAY$ and $\triangle NAM$ are also similar, since M is the midpoint of AE. Thus $\angle ANZ = \angle ABZ = \angle ABY = \angle ANM$, implying that N, M, Z are collinear. Now we have $\angle ZMD = 90° - \angle ZMA = \angle EAZ = \angle ZED$ (the last equality because ED is tangent to ω); hence $ZMED$ is a cyclic quadrilateral. It follows that $\angle ZDM = \angle ZEA = \angle ZAD$, which is enough to conclude that MD is tangent to the circumcircle of AZD.

   Remark. The statement remains valid if $\angle B \geq \angle C$.