Berkeley Math Circle
Monthly Contest 4 – Solutions

1. There are 10 bags full of coins. All coins look the same and all wight 10 grams, except the coins from one bag that are fake and all weight 9 grams. Given a scale, how could you tell which bag has the wrong coins in just one measurement? Explain your answer!

   Solution. Take one coin from the first bag, 2 from the second, 3 from the third, $\dots$, 10 from the 10th and place them on the scale. The total mass shown by the scale should be $(1+2+\cdots+10)\cdot 10 - x$ grams, where x is the number of fake coins (each coin weighted 10 grams, except for the fake ones that are one gram lighter). Since the scale will show $550 - x$ grams we can see how many coins are fake and that is enough to deduce from which bag those coines were taken.

2. Find all prime numbers p such that $p^2 + 2007p - 1$ is prime as well.

   Hint. Except for 3, prime numbers are not divisible by 3. Hence if p is not equal to 3 then either $p = 3k+1$ or $p = 3k-1$ for some integer k. If you wish you may use lists of prime numbers from the internet (e.g. www.imomath.com/primes)

   Solution. If $p = 3$, then $p^2 + 2007p - 1 = 6029$ which is a prime. For $p \neq 3$, we know that $p = 3k \pm 1$ hence $p^2 + 2007p - 1 = 9k^2 \pm 6k + 1 + 2007p - 1 = 9k^2 \pm 6k + 2007p$ which is divisible by 3 and can't be prime. Thus the only such prime number is $p = 3$.

3. The sequence of numbers $1, 2, 3, \dots, 100$ is written on the blackboard. Between each two consecutive numbers a square box is drawn. Player A starts the game and the players A and B alternate the moves. In each turn a player choses an empty box and places "+" or "·" sign in it. After all the boxes are filled the expression on the blackboard is evaluated and if the result is an odd number the winner is A. Otherwise the winner is B. Determine which of the players has a winning strategy and what the strategy is.

   Solution. A has the winning strategy. She should first place the sign "+" between the numbers 1 and 2. After that she should group the square boxes into pairs: each pair consisting of two boxes adjacent to the same odd number. Then the player A should make sure that there is at least one sign "·" in each pair of boxes. Since $\cdot$ has the priority over $+$, after all products of numbers are calculated, 1 will be the only odd summand in the whole expression, hence the sum is odd. A can achieve this goal by placing $\cdot$ sign in the box of the pair where B has previously put his sign.

4. The sum of the squares of five real numbers a_1, a_2, a_3, a_4, a_5 equals 1. Prove that the least of the numbers $(a_i - a_j)^2$, where $i, j = 1, 2, 3, 4, 5$ and $i \neq j$, does not exceed $1/10$.

   Solution. Assume w.l.o.g. that $a_1 \le a_2 \le a_3 \le a_4 \le a_5$. If m is the least value of $|a_i - a_j|$, $i \neq j$, then $a_{i+1} - a_i \ge m$ for $i = 1, 2, \dots, 5$, and consequently $a_i - a_j \ge (i-j)m$ for any $i, j \in \{1, \dots, 5\}$, $i > j$. Then it follows that

   $$\sum_{i>j} (a_i - a_j)^2 \ge m^2 \sum_{i>j} (i-j)^2 = 50m^2.$$

   On the other hand, by the condition of the problem,

   $$\sum_{i>j} (a_i - a_j)^2 = 5 \sum_{i=1}^{5} a_i^2 - (a_1 + \cdots + a_5)^2 \le 5.$$

   Therefore $50m^2 \le 5$; i.e., $m^2 \le \frac{1}{10}$.

5. Let $ABCD$ be a parallelogram. A variable line l passing through the point A intersects the rays BC and DC at points X and Y, respectively. Let K and L be the centers of the excircles of triangles ABX and ADY, touching the sides BX and DY, respectively. Prove that the size of angle KCL does not depend on the choice of the line l.

   Solution. Since $\angle ADL = \angle KBA = 180^\circ - \frac{1}{2}\angle BCD$ and $\angle ALD = \frac{1}{2}\angle AYD = \angle KAB$, triangles ABK and LDA are similar. Thus $\frac{BK}{BC} = \frac{BK}{AD} = \frac{AB}{DL} = \frac{DC}{DL}$, which together with $\angle LDC = \angle CBK$ gives us $\triangle LDC \sim \triangle CBK$. Therefore $\angle KCL = 360^\circ - \angle BCD - (\angle LCD + \angle KCB) = 360^\circ - \angle BCD - (\angle CKB + \angle KCB) = 180^\circ - \angle CBK$, which is constant.