Berkeley Math Circle
Monthly Contest 3 – Solutions

1. Given 8 oranges on the table, 7 of them have exactly the same weight and the 8th is a little bit lighter. You are given a balance that can measure oranges against each other and you are allowed to use the balance at most twice! How can you determine which one of the oranges is lighter then the others? Explain your answer!

   Remark. All oranges look the same and the difference in the weight of the lighter orange is not big enough for you to distinguish it without using the balance. The balance doesn't have any weights or numbers. If you put some oranges on each side of the balance, you can only tell which side (if any) is heavier.

   Solution. First we put 3 oranges on the left and 3 on the right-hand side of the balance. In the case that balance shows equal weights, one of the remaining two oranges is lighter and in the second measurment we can easily see which one. However if one side (say left) is lighter, then we know that the orange we are looking for is one of the three that appear on the left side. Let us denote these oranges by A, B, C. In the second measurment we can measure A against B. If one of them is lighter than the other, we are done (as the lighter orange is the one we are looking for). If the weights of A and B are the same, then C is the lighter orange.

2. Find all prime numbers p such that $p^2 + 8$ is prime number, as well.

   Remark. A number p is prime if it has exactly 2 divisors: 1 and p. Numbers $2, 3, 5, 7, 11, 13, \dots$ are prime, while 4 and 2007 are not.

   Hint. Write down first several prime numbers (hint - you can copy them from the paragraph above), calculate $p^2 + 8$ for them, and look at those that happen to be composite. Notice further that they all have a common divisor.

   Solution. For $p = 3$ we have $p^2 + 8 = 17$, which is prime. If $p \neq 3$ then p is not divisible by 3. The remainder of p when divided by 3 is either 1 or 2. This means that $p = 3k + 1$ for some integer k, or $p = 3l + 1$ for some integer l. In the first case we get $p^2 = 9k^2 + 6k + 1$ and in the second, $p^2 = 9l^2 - 6l + 1$. In both cases p^2 gives a remainder 1 upon division by 3. Hence $p^2 + 8$ is divisible by 3 for all prime numbers different than 3.

3. p is a prime number such that the period of its decimal reciprocal is 200. That is,

   $$\frac{1}{p} = 0.XXXX\dots$$

   for some block of 200 digits X, but

   $$\frac{1}{p} \neq 0.YYYY\dots$$

   for all blocks Y with less than 200 digits. Find the 101st digit, counting from the left, of X.

   Solution. Let X be a block of n digits and let $a = 0.X\dots$. Then $10^n a = X.X\dots$. Subtracting the previous two equalities gives us $(10^n - 1)\,a = X$, i.e. $a = \frac{X}{10^n - 1}$.

   Then the condition that $a = \frac{1}{p}$ reduces to $\frac{1}{p} = \frac{X}{10^n - 1}$ or $pX = 10^n - 1$. For a given p and n, such an X exists if and only if p divides $10^n - 1$. Thus p divides $10^{200} - 1$ but not $10^n - 1$, $1 \le n \le 199$. Note that $10^{200} - 1$ can be factored in this way:

   $$\begin{aligned} 10^{200} - 1 &= \left(10^{100}\right)^2 - 1 \\ &= \left(10^{100} - 1\right)\left(10^{100} + 1\right). \end{aligned}$$

   Since p is prime and does not divide $10^{100} - 1$, it must divide $10^{100} + 1$, so that $10^{100} + 1 = kp$ for an integer k and $X = \frac{\left(10^{100}-1\right)\left(10^{100}+1\right)}{p} = \left(10^{100} - 1\right)k$.

   If $p = 2$, 3, 5, or 7, the fraction $\frac{1}{p}$ either terminates or repeats less than 200 digits. Therefore $p > 10$ and $k < \frac{10^{100}}{p} < 10^{99}$. Now let us calculate the 101st digit of $X = 10^{100}k - k$, i.e. the digit representing multiples of 10^{99}. Since $10^{100}k$ is divisible by 10^{100}, its 10^{99}s digit and all later digits are 0. Since $k < 10^{99}$, k does not contribute a digit to the 10^{99}s place, but it generates a *borrow* to this place, changing it into a 9. Thus the 101st digit of X is a 9.

4. Let $ABCD$ be a trapezoid such that $AB \| CD$ and let P be the point on the extension of the diagonal AC such that C is between A and P. If X and Y are midpoints of the segments AB and CD, and M, N intersection points of the lines PX, PY with BC, DA (respectively) prove that MN is parallel to AB.

Solution. Notice that $\frac{BM}{MC} = \frac{S_{\Delta PMB}}{S_{\Delta PMC}}$. Since $S_{\Delta PMB} + S_{\Delta XMB} = S_{\Delta PXB} = S_{\Delta PXA} = S_{\Delta PCM} + S_{\Delta ACM} + S_{\Delta AXM}$, we get that $S_{\Delta PMB} = S_{\Delta PMC} + S_{\Delta ACM}$. Hence,

$$\frac{BM}{MC} = \frac{S_{\Delta PMC} + S_{\Delta ACM}}{S_{\Delta PMC}} = 1 + \frac{S_{\Delta ACM}}{S_{\Delta PMC}} = 1 + \frac{AC}{CP}. \tag{1}$$

Similarly as above we will use that $\frac{AN}{ND} = \frac{S_{\Delta PNA}}{S_{\Delta PDN}}$. Since Y is the midpoint of CD, we have: $S_{\Delta PNC} = S_{\Delta PYC} + S_{\Delta YCN} = S_{\Delta PDY} + S_{\Delta DYN} = S_{\Delta PDN}$. From $S_{\Delta PNA} = S_{\Delta PNC} + S_{\Delta CNA}$ we get

$$\frac{AN}{ND} = \frac{S_{\Delta PNA}}{S_{\Delta PDN}} = \frac{S_{\Delta PNC} + S_{\Delta CNA}}{S_{\Delta PNC}} = 1 + \frac{S_{\Delta CNA}}{S_{\Delta PNC}} = 1 + \frac{AC}{CP}. \tag{2}$$

From (1) and (2) it follows that $\frac{AN}{ND} = \frac{MB}{MC}$, which implies $NM \| AB \| CD$.

5. Let $0 < a_0 \leq a_1 \leq \cdots \leq a_n$. If z is a complex number such that $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ prove that $|z| \geq 1$.

   Solution. Assume that $|z| < 1$. If $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ then $a_0 z^{n+1} + a_1 z^n + \cdots + a_n z = 0$ and subtracting these two equations leads to $a_0 z^{n+1} + (a_1 - a_0) z^n + \cdots + (a_n - a_{n-1}) z - a_n = 0$, or equivalently $a_n = a_0 z^{n+1} + (a_1 - a_0) z^n + \cdots + (a_n - a_{n-1}) z$ hence

$$\begin{aligned} |a_n| &= |a_0 z^{n+1} + (a_1 - a_0) z^n + \cdots + (a_n - a_{n-1}) z| \\ &\leq a_0 |z|^{n+1} + (a_1 - a_0)|z|^n + \cdots + (a_n - a_{n-1})|z| \\ &< a_0 + (a_1 - a_0) + \cdots + (a_n - a_{n-1}) = a_n, \end{aligned}$$

   which is impossible. Thus $|z| \geq 1$.