Berkeley Math Circle
Monthly Contest 3
Due December 4, 2007

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 3
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Given 8 oranges on the table, 7 of them have exactly the same weight and the 8th is a little bit lighter. You are given a balance that can measure oranges against each other and you are allowed to use the balance at most twice! How can you determine which one of the oranges is lighter then the others? Explain your answer!

   Remark. All oranges look the same and the difference in the weight of the lighter orange is not big enough for you to distinguish it without using the balance. The balance doesn't have any weights or numbers. If you put some oranges on each side of the balance, you can only tell which side (if any) is heavier.

2. Find all prime numbers p such that $p^2 + 8$ is prime number, as well.

   Remark. A number p is prime if it has exactly 2 divisors: 1 and p. Numbers $2, 3, 5, 7, 11, 13, \dots$ are prime, while 4 and 2007 are not.

   Hint. Write down first several prime numbers (hint - you can copy them from the paragraph above), calculate $p^2 + 8$ for them, and look at those that happen to be composite. Notice further that they all have a common divisor.

3. p is a prime number such that the period of its decimal reciprocal is 200. That is,

   $$\frac{1}{p} = 0.XXXX\dots$$

   for some block of 200 digits X, but

   $$\frac{1}{p} \neq 0.YYYY\dots$$

   for all blocks Y with less than 200 digits. Find the 101st digit, counting from the left, of X.

4. Let $ABCD$ be a trapezoid such that $AB \| CD$ and let P be the point on the extension of the diagonal AC such that C is between A and P. If X and Y are midpoints of the segments AB and CD, and M, N intersection points of the lines PX, PY with BC, DA (respectively) prove that MN is parallel to AB.

5. Let $0 < a_0 \leq a_1 \leq \cdots \leq a_n$. If z is a complex number such that $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ prove that $|z| \geq 1$.