Berkeley Math Circle
Monthly Contest 2 – Solutions

1. Four friends, *One*, *Two*, *Five*, and *Ten* are located on one side of the dark tunnel, and have only one flashlight. It takes one minute for person One to walk through the tunnel, two minutes for Two, five for Five, and ten for Ten. The tunnel is narrow and at most two people can walk at the same time with the flashlight. Whenever two people walk together they walk at the speed of the slower one. Show that all four friends can go from one side of the tunnel to the other one in 17 minutes.

   Remark. Your explanation should be something like this: The friends X and Y first go through the tunnel using the flashlight, then X returns with the flashlight to the other side,...

   Solution. Friends One and Two should walk to the other side. It will take them **2** minutes. Then, One returns - this will last additional **1** minute. Then, Ten and Five go to the other side – this will take **10** minutes, but Two should return the flashlight (**2** additional minutes), and together with One, go through the tunnel for the last time – additional **2** minutes. It remains to notice that the time spent is exactly 17 minutes.

2. The integers from 1 to 16 are arranged in a 4×4 array so that each row, column and diagonal adds up to the same number.

   (a) Prove that this number is 34.

   (b) Prove that the four corners also add up to 34.

   Solution.

   (a) Add up all the numbers in the square in two ways. On the one hand, it consists of four rows, each adding to the common sum S, so the entire square adds to $4S$. But the numbers in the square are also the integers from 1 to 16, whose sum is

   $$\frac{16 \cdot 17}{2} = 152.$$

   Hence

   $$\text{Sum of all numbers in square} = 4S = 152$$

   whence $S = 34$.

   (b) Denote the cells of the table as shown below.

a	b	c	d
e	f	g	h
i	j	k	m
n	p	q	r

   Adding the equations

   $$\begin{aligned} a+b+c+d &= 34 \\ n+p+q+r &= 34 \\ a+f+k+r &= 34 \\ n+j+g+d &= 34 \\ -b-f-j-p &= -34 \\ -c-g-k-q &= -34 \end{aligned}$$

   yields $2a + 2d + 2n + 2r = 68$, or $a + d + n + r = 34$.

3. Let A_1, B_1, C_1 be the points on the sides BC, CA, AB (respectively) of the triangle ABC. Prove that the three circles circumscribed about the triangles $\triangle AB_1C_1$, $\triangle BC_1A_1$, and $\triangle CA_1B_1$ intersect at one point.

   Solution. Denote by α, β, and γ the angles of $\triangle ABC$. Assume that the circumcircles of $\triangle AB_1C_1$ and BC_1A_1 intersect at the point M. Assume that M is in the interior of $\triangle ABC$ (the other cases are similar). By the properties of the inscribed quadrilaterals AB_1MC_1 and BA_1MC_1 we get: $\angle C_1MB_1 = 180° - \angle C_1AB_1 = 180° - \alpha$ and $\angle C_1MA_1 = 180° - \angle C_1BA_1 = 180° - \beta$ hence $\angle B_1MA_1 = 360° - (\angle C_1MB_1 + \angle C_1MA_1) = 360° - [360° - (\alpha + \beta)] = \alpha + \beta$. Therefore $\angle B_1MA_1 + \angle B_1CA_1 = \angle B_1MA_1 + \gamma = \alpha + \beta + \gamma = 180°$. Thus the points B_1, M, A_1, and C belong to a circle.

4. A Mystic Four Calculator has a four-digit display and four buttons. The calculator works as follows: Pressing button 1 *replaces* the number in the display with 1; Pressing button 2 *divides* the number in the display by 2; Pressing button 3 *subtracts* 3 from the number in the display; Pressing button 4 *multiplies* the number in the display by 4.

   Initially the display shows 0. Any operation yielding a negative, fractional, or five-digit answer is ignored.

   (a) Can 2007 appear in the display?

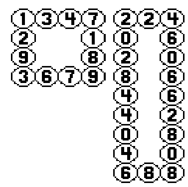
   (b) Can 2008 appear in the display?

   Solution.

   (a) No. Notice that if the number on the display is not divisible by 3, then none of the operation can have as a result a number divisible by 3. At the start, 1 is the only button producing a result, so we are required to press it at some point. After that the number will never be divisible by 3 again, and 2007 is divisible by 3.

   (b) Yes. For instance, press 1 so that the display shows 1, press 4 six times so that the display shows 4096, and press 3 696 times so that the display shows 2008.

5. A *number bracelet in base* m is made by choosing two non-negative integers less than m (not both 0) and continuing in a clockwise loop, each succeeding number being the mod m sum of its two predecessors. The figure is closed up as soon as it starts to repeat. The figure to the right shows two number bracelets in base 10, starting with the pairs (1,3), and (2,2), respectively. Prove that the lengths of all number bracelets in a given base are divisors of the length of the number bracelet beginning with $(0, 1)$.



Solution. The elements of a number bracelet N will be denoted $N_0, N_1, N_2, \ldots$; N_0 and N_1 being the starting numbers. Let F be the bracelet with starting numbers $F_0 = 0$, $F_1 = 1$. All congruences are modulo m unless otherwise noted.
Since there are only m^2 possible pairs (N_k, N_{k+1}), there must be repetition, so that

$$(N_k, N_{k+1}) = (N_{p+k}, N_{p+k+1}) \tag{1}$$

for some $p \geq 1$ and k. If this is true for one k, it must be true for the next k, since $N_{k+1} = N_{p+k+1}$ by hypothesis and $N_{k+2} \equiv N_k + N_{k+1} = N_{p+k} + N_{p+k+1} \equiv N_{p+k+2}$, and so it is true for all k bigger than that k. It must also be true for the previous k, since $N_k = N_{p+k}$ by hypothesis and $N_{k-1} \equiv N_{k+1} - N_k = N_{p+k+1} - Np + k \equiv N_{p+k-1}$, and so it is true for all k smaller than that k. Therefore (1) is true for all k. Let P_N, the *period* of N, be the smallest p such that (1) is true for some k (and therefore for all k).

It is clear that if $0 \leq i, j < P(N)$, $i \neq j$, then $(N_i, N_{i+1}) \neq (N_j, N_j + 1)$. Therefore, among the terms N_0 through N_{P_N}, there is no repetition even of pairs of adjacent terms. But because of property (1) of P_N, $(N_0, N_1) = (N_{P_N}), (N_{P_N+1})$ and the repetition will continue. Thus the number bracelet N consists of a single loop of P_N elements.

We will prove by induction that for all $n \geq 1$,

$$N_n \equiv N_0 F_{n-1} + N_1 F_n. \tag{2}$$

The case $n = 1$ is trivial, and $n = 2$ follows directly from the definition of N. If (2) is true for $n = k$ and $n = k - 1$, $N_{k+1} \equiv N_k + N_{k-1} \equiv N_0 F_{k-1} + N_1 F_k + N_0 F_{k-2} + N_1 F_{k-1} = N_0 F_k + N_1 F_{k+1}$ so it is also true for $n = k + 1$.
It is clear from the circularity of the number bracelet that $N_{k+R} = N_k$ for all $k > 1$ if and only if $P_N | R$. Since $N_{k+P_F} = N_0 F_{P_F+k-1} + N_1 F_{P_F+k} = N_0 F_{k-1} + N_1 F_k = N_k$ for all $k > 1$, $P_N | P_F$. This completes the proof.