Berkeley Math Circle
Monthly Contest 1
Due October 9, 2007

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 1
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

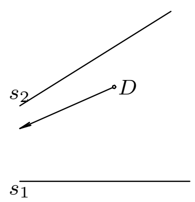
Enjoy solving these problems and good luck!

Problems

1. If a and b are positive integers prove that
$$a + b \leq 1 + ab.$$

2. A man has three pets: a mouse, a cat, and a dog. If the man leaves the cat and the dog alone, then the dog would kill the cat. If the man leaves the cat and the mouse alone, the cat would eat the mouse. One day the man decided to take his animals to the other side of the river. However, he has a small boat in which he can fit only one of the animals at a time. Show that with multiple trips of the boat, it is possible for the man to take all of the animals to the other side of the river safely.

3. On a small piece of paper two line segments s_1 and s_2 are drawn as shown on the picture. The extensions of s_1 and s_2 eventually intersect at a point P that doesn't belong to the piece of paper. If D is an arbitrary point marked on the paper, show how to construct a segment of the line connecting D and P using just a straight edge and a compass and performing all constructions on the given piece of paper.



4. Denote by $f(n)$ the integer obtained by reversing the digits of a positive integer n. Find the greatest integer that is certain to divide $n^4 - f(n)^4$ regardless of the choice of n.

5. Given a polynomial $P(x)$ with integer coefficients, assume that for every positive integer n we have $P(n) > n$. Consider the sequence
$$x_1 = 1,\ x_2 = P(x_1),\ \ldots, x_n = P(x_{n-1}),\ \ldots$$
If for every positive integer N there exists a member of the sequence divisible by N, prove that $P(x) = x + 1$.