Berkeley Math Circle
Monthly Contest 8
Due April 29, 2008

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 8
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Prove that it is possible to put $+$ or $-$ instead each of the 99 symbols $\star$ in the expression

$$1 \star 2 \star 3 \star 4 \star \cdots \star 98 \star 99 \star 100$$

   so that the result of the expression is 0.

   Hint: Try first with smaller expressions, like $1 \star 2 \star 3 \star 4$, or $1 \star 2 \star 3 \star \cdots \star 8$.

2. Do there exist different positive integers x, y, and z such that $x^x + y^y = z^z$?

3. A rectangular board is tiled with dominoes and trominoes (of either shape). An ant starts at any point in one of the cracks between the tiles. The ant can travel in any of the four directions, but once it chooses a direction, it continues in that direction until it bumps into a tile or reaches the perimeter of the board. Find the smallest number of trominoes necessary to trap the ant, i.e. for at least one initial position, it is unable to leave the board.

4. Assume that all the angles of a given triangle ABC are smaller than $120°$. Equilateral triangles AFB, BDC and CEA are constructed in the exterior of $\triangle ABC$. Prove that the lines AD, BE, and CF pass through one point S for which $SD + SE + SF = 2(SA + SB + SC)$.

5. There is a row of real numbers, infinite in both directions. The square of each number is 1 more than the product of the numbers on either side. Prove that if four adjacent numbers are integers, then at least one number in the row is 0.