Berkeley Math Circle
Monthly Contest 6
Due March 4, 2008

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 6
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Ten different points are marked on a circle. Two players A and B play the following game. A moves first and the players alternate their moves. In each of the moves a player connects two of the points with a straight line segment. A player whose segment crosses a segment previously drawn will lose the game. Which player has a winning strategy and what is the strategy.

2. Prove that no integer greater than 2008 can be equal to the sum of squares of its digits.

3. If $x \geq 4$ is a real number prove that
$$\sqrt{x} - \sqrt{x-1} \geq \frac{1}{x}.$$

4. Wally has a very unusual combination lock number. It has five digits, all different, and is divisible by 111. If he removes the middle digit and replaces it at the end, the result is a larger number that is still divisible by 111. If he removes the digit that is now in the middle and replaces it at the end, the result is a still larger number that is still divisible by 111. What is Wally's combination lock number? Explain your answer!

5. Let $A_0, A_1, \dots, A_n$ be points in a plane such that

   (i) $A_0A_1 \leq \frac{1}{2}A_1A_2 \leq \cdots \leq \frac{1}{2^{n-1}}A_{n-1}A_n$ and

   (ii) $0 < \measuredangle A_0A_1A_2 < \measuredangle A_1A_2A_3 < \cdots < \measuredangle A_{n-2}A_{n-1}A_n < 180^\circ$,

   where all these angles have the same orientation. Prove that the segments A_kA_{k+1}, A_mA_{m+1} do not intersect for each k and n such that $0 \leq k \leq m-2 < n-2$.