Berkeley Math Circle
Monthly Contest 5
Due February 5, 2008

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 5
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. The rectangle $MNPQ$ is inside the rectangle $ABCD$. The portion of the rectangle $ABCD$ outside of $MNPQ$ is colored in green. Using just a straightedge construct a line that divides the green figure in two parts of equal areas.

2. Determine the positive real numbers a and b satisfying $9a^2 + 16b^2 = 25$ such that $a \cdot b$ is maximal. What is the maximum of $a \cdot b$? Explain your answer!

   Hint. If x and y are any two real numbers then $x^2 + y^2 \geq 2xy$.

3. Find all pairs of integers (x, y) for which $x^2 + xy = y^2$.

4. Let n be a positive integer. Prove that there exist distinct positive integers x, y, z such that

$$x^{n-1} + y^n = z^{n+1}.$$

5. Let ABC be a triangle such that $\angle A = 90^\circ$ and $\angle B < \angle C$. The tangent at A to its circumcircle ω meets the line BC at D. Let E be the reflection of A across BC, X the foot of the perpendicular from A to BE, and Y the midpoint of AX. Let the line BY meet ω again at Z. Prove that the line BD is tangent to the circumcircle of triangle ADZ.