Berkeley Math Circle

Monthly Contest 2
Due November 6, 2007

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 2
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Four friends, *One*, *Two*, *Five*, and *Ten* are located on one side of the dark tunnel, and have only one flashlight. It takes one minute for person One to walk through the tunnel, two minutes for Two, five for Five, and ten for Ten. The tunnel is narrow and at most two people can walk at the same time with the flashlight. Whenever two people walk together they walk at the speed of the slower one. Show that all four friends can go from one side of the tunnel to the other one in 17 minutes.

   Remark. Your explanation should be something like this: The friends X and Y first go through the tunnel using the flashlight, then X returns with the flashlight to the other side,...

2. The integers from 1 to 16 are arranged in a 4×4 array so that each row, column and diagonal adds up to the same number. Prove that this number is 34, and that the four corners also add up to 34.

3. Let A_1, B_1, C_1 be the points on the sides BC, CA, AB (respectively) of the triangle ABC. Prove that the three circles circumscribed about the triangles $\triangle AB_1C_1$, $\triangle BC_1A_1$, and $\triangle CA_1B_1$ intersect at one point.

4. A Mystic Four Calculator has a four-digit display and four buttons. The calculator works as follows: Pressing button 1 *replaces* the number in the display with 1; Pressing button 2 *divides* the number in the display by 2; Pressing button 3 *subtracts* 3 from the number in the display; Pressing button 4 *multiplies* the number in the display by 4.

   Initially the display shows 0. Any operation yielding a negative, fractional, or five-digit answer is ignored.

   (a) Can 2007 appear in the display?

   (b) Can 2008 appear in the display?

5. A *number bracelet in base* m is made by choosing two non-negative integers less than m (not both 0) and continuing in a clockwise loop, each succeeding number being the mod m sum of its two predecessors. The figure is closed up as soon as it starts to repeat. The figure to the right shows two number bracelets in base 10, starting with the pairs (1,3), and (2,2), respectively. Prove that the lengths of all number bracelets in a given base are divisors of the length of the number bracelet beginning with $(0, 1)$.