Berkeley Math Circle
Monthly Contest 1 – Solutions

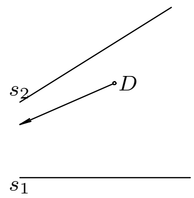
1. If a and b are positive integers prove that
$$a + b \le 1 + ab.$$

   Solution. Since a and b are positive integers we know that $a \ge 1$ and $b \ge 1$. This implies that $(a-1)(b-1) \ge 0$ hence $ab - a - b + 1 \ge 0$ which is equivalent to the given inequality.

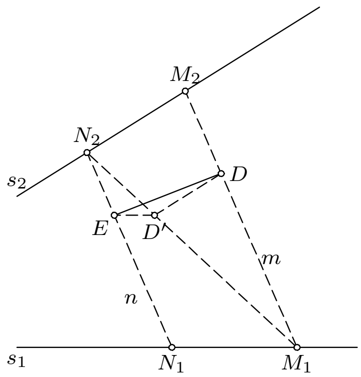
2. A man has three pets: a mouse, a cat, and a dog. If the man leaves the cat and the dog alone, then the dog would kill the cat. If the man leaves the cat and the mouse alone, the cat would eat the mouse. One day the man decided to take his animals to the other side of the river. However, he has a small boat in which he can fit only one of the animals at a time. Show that it is possible for the man to take all of his animals to the other side of the river safely.

   Solution. Let A be the side of the river where the trip starts, and B the other side. In the first trip the man should take the cat to the side B. He would leave the mouse and the dog together, but that is not dangerous. Then he will return back for a dog and take it to the side B. However when he brings the dog, then he returns the cat from B to A. Then he takes the mouse from A to B and comes back to the side A to finally take the cat.

3. On a small piece of paper two line segments s_1 and s_2 are drawn as shown on the picture. The extensions of s_1 and s_2 eventually intersect at a point P that doesn't belong to the piece of paper. If D is an arbitrary point marked on the paper, show how to construct a segment of the line connecting D and P using just a straight edge and a compass and performing all constructions on the given piece of paper.



   Solution. Let m be an arbitrary line through D that intersects s_1 and s_2 in M_1 and M_2 and let n be an arbitrary line parallel to m that intersects s_1 and s_2 at N_1 and N_2. If we find a point E on n such that $M_1D : DM_2 = N_1E : EN_2$ then we will be sure that s_1, s_2, and ED pass through the same point (because of the Thales' theorem). In order to construct E we first construct the point D' on M_1N_2 such that $DD' \| s_2$. Then $M_1D : DM_2 = M_1D' : D'N_2$ because of the similarity of the triangles M_1DD' and $M_1M_2N_2$. Similarly, E is the point of n such that $ED' \| s_1$ (now the similar triangles are N_2ED' and $N_2N_1M_1$).



4. Denote by $f(n)$ the integer obtained by reversing the digits of a positive integer n. Find the greatest integer that is certain to divide $n^4 - f(n)^4$ regardless of the choice of n.

   Solution. The answer is 99. Let $x = \overline{d_{n-1} \dots d_2 d_1 d_0}$, i.e. $x = 10^{n-1}d_{n-1} + \cdots + 10^2 d_2 + 10 d_1 + d_0$. Then $y = f(x) = 10^{n-1}d_0 + 10^{n-2}d_1 + \cdots + 10 d_{n-2} + d_{n-1}$.

   Let us show separately that $9 | x^4 - y^4$ and that $11 | x^4 - y^4$. Since $10 \equiv 1 \pmod 9$, $x \equiv y \equiv d_{n-1} + d_{n-2} + \cdots + d_1 + d_0 \pmod 9$, so $9 | x - y$ and therefore also $9 | (x-y)(x^3 + x^2 y + x y^2 + y^3) = x^4 - y^4$. Since $10 \equiv -1 \pmod{11}$, $x \equiv d_{n-1} - d_{n-2} + \cdots \mp d_1 \pm d_0 \pmod{11}$ and $y \equiv d_0 + d_1 + \cdots \mp d_{n-2} \pm d_{n-1} \pmod{11}$, i.e. $x \equiv \pm y$. In either case $x^4 \equiv y^4$ mod 11, so $11 | x^4 - y^4$. Since 9 and 11 are relatively prime, their product 99 must divide $x^4 - y^4$.

   To see that there is no larger integer that always divides $n^4 - f(n)^4$, let $n = 10$, so that $f(n) = 01 = 1$, $n^4 - f(n)^4 = 9999 = 3^2 \cdot 11 \cdot 101$, and let $n = 21$, so that $f(n) = 12$ and $n^4 - f(n)^4 = 173745 = 3^5 \cdot 5 \cdot 11 \cdot 13$. The greatest common divisor of these two values of $n^4 - f(n)^4$ is $3^2 \cdot 11 = 99$, so there is no larger integer certain to divide $n^4 - f(n)^4$.

   Remark. From the solution it is easy to see that 99 is even the common divisor of the numbers of the form $n^2 - f(n)^2$.

5. Given a polynomial $P(x)$ with integer coefficients, assume that for every positive integer n we have $P(n) > n$. Consider the sequence
$$x_1 = 1,\ x_2 = P(x_1),\ \dots,\ x_n = P(x_{n-1}),\ \dots$$

If for every positive integer N there exists a member of the sequence divisible by N, prove that $P(x) = x + 1$.

Solution. Assume the contrary. The polynomial $Q(x) = P(x) - x$ is non-decreasing for all x greater than some M, as otherwise it wouldn't be positive, and it has to be bigger than 1 for each n.
If there are infinitely many integers n such that $P(n) = n + 1$ then we would have $P(x) = x + 1$ (nonzero polynomial $P(x) - x - 1$ can have only finitely many zeroes). Therefore there exists an index $k \in \mathbb{N}$ such that $x_k > M$ and $P(x_l) - x_l \geq 2$ for $l \geq k$.
Assume that $d = x_{l+1} - x_l \geq 2$. Then d is a divisor of $P(x_{l+1}) - P(x_l)$ i.e. of $x_{l+2} - x_{l+1}$. Since, if $a \equiv b \pmod{d}$, then $P(a) \equiv P(b)$, by induction all the numbers $x_l, x_{l+1}, \dots$ have the same remainder modulo d. If that remainder is not zero than no term of the sequence could be divisible by d^m for sufficiently large m which would contradict the assumptions.
Hence x_l is divisible by $x_{l+1} - x_l$ for all $l \geq k$. Let $x_l = c_l(P(x_l) - x_l)$. Then $P(x_l) = \frac{c_l+1}{c_l} x_l$ holds for infinitely many pairs of integers (x_l, c_l) such that $x_l \to \infty$. Because $\frac{c_l+1}{c_l} x_l = \left(1 + \frac{1}{c_l}\right) x_l \leq 2x_l$, the degree of P cannot exceed 1 so that either $P(x) = 2x$ or $P(x) = x$. Neither of these polynomials satisfy the given condition, and this is a contradiction.