Berkeley Math Circle
Monthly Contest 8
Due April 24, 2007

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 8
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Define
$$A = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \ddots + \cfrac{1}{2006 + \cfrac{1}{2007}}}}} \quad \text{and} \quad B = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \ddots + \cfrac{1}{2005 + \cfrac{1}{2006}}}}}.$$
Which of the two numbers is greater, A or B? Explain your answer!

2. A group of mathematicians is lost in a forest. The forest has a shape of a strip that is infinitely long and 1 mile wide. Prove that they can choose a path that will guarantee them a way out and that is at most $2\sqrt{2}$ miles long.

3. The numbers $1, 8, 4, 0$ are the first four terms of an infinite sequence. Every subsequent term is obtained as the last digit of the sum of previous four terms. Therefore the fifth term of the sequence is 3 because $1 + 8 + 4 + 0 = 13$; the sixth term is 5 because $8 + 4 + 0 + 3 = 5$, and so on.

   (a) Will $2, 0, 0, 7$ ever appear as a subsequence?

   (b) Will $1, 8, 4, 0$ appear again as a subsequence?

   Explain your answer!

4. Let X, Y, and Z be the points on the sides BC, CA, and AB of the triangle ABC, such that $\triangle XYZ \sim \triangle ABC$ ($\sphericalangle X = \sphericalangle A$, $\sphericalangle Y = \sphericalangle B$). Prove that the orthocenter of $\triangle XYZ$ coincides with the circumcenter of $\triangle ABC$.

5. Let $n > 1$ be an odd integer. Prove that every integer l satisfying $1 \leq l \leq n$ can be represented as a sum or difference of two integers each of which is less than n and relatively prime to n.