Berkeley Math Circle
Monthly Contest 6
Due March 6, 2007

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 6
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. If a and b are two positive numbers not greater than 1 prove that
$$\frac{a+b}{1+ab} \leq \frac{1}{1+a} + \frac{1}{1+b}.$$
When does the equality hold?

2. A car is moving at a constant speed. Every 15 minutes it makes a 90° turn to either left or right. If the car has started the trip at some point A, prove that it can return to A only after an integer number of hours.

3. Let M be an interior point of a parallelogram $ABCD$. Prove that $MA + MB + MC + MD$ is strictly less than the length of the perimeter of $ABCD$.

4. Prove that the product of 6 consecutive positive integer is never equal to n^5 for some positive integer n.

5. The point K lies in the interior of the unit circle. Four lines are drawn through K such that each two adjacent lines form an angle of 45°, as shown in the picture. In such a way the circle is divided into 8 regions. Four of these regions are colored such that no two colored regions share more than one point. Find the maximal and minimal possible value for the total area of colored regions.