Berkeley Math Circle
Monthly Contest 4
Due January 9, 2007

Instructions
This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 4
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Do there exist positive integers x, y, and z such that $x^{2006} + y^{2006} = z^{2007}$? Explain your answer.

   Remark. If the answer is *yes*, you should give an example of such x, y, and z. If the answer is *no*, you should prove that no x, y, z can satisfy the above equation.

2. Let O be the intersection of the diagonals AC and BD of the convex quadrilateral $ABCD$. Let S_1, S_2, S_3, and S_4 denote the areas of the triangles ABO, BCO, CDO, and DAO.

   (a) Prove that $S_1 \cdot S_3 = S_2 \cdot S_4$.

   (b) Does there exist a quadrilateral $ABCD$ such that S_1, S_2, S_3, and S_4 are consecutive positive integers in some order?

3. 2006 vertices of a regular 2007-gon are red. The remaining vertex is green. Let G be the total number of polygons whose one vertex is green and the others are red. Denote by R the number of polygons whose all vertices are red. Which number is bigger, R or G? Explain your answer.

4. A sequence of numbers $\{a_n\}$ is given by $a_1 = 1$, $a_{n+1} = 2a_n + \sqrt{3a_n^2 + 1}$ for $n \geq 1$. Prove that each term of the sequence is an integer.

5. A finite set of circles in the plane is called *nice* if it satisfies the following three conditions:

   (i) No two circles intersect in more than one point;

   (ii) For every point A of the plane there are at most two circles passing through A;

   (iii) Each circle from the set is tangent to exactly 5 other circles form the set.

   Does there exist a nice set consisting of exactly

   (a) 2006 circles?

   (b) 2007 circles?