Berkeley Math Circle
Monthly Contest 7 – Solutions

1. Find all positive integers n such that $n(n+1)$ is a perfect square.

   Solution. Since n and $n+1$ are coprime numbers, if $n(n+1)$ is a perfect square, then each of n and $n+1$ has to be a perfect square itself. However that is impossible since if $n = x^2$ and $n+1 = y^2$ we would have $1 = y^2 - x^2 = (y-x)(y+x)$ and 1 can't be expressed as a product of two different integers. Hence there are no such integers.

2. Prove that
$$A = \sqrt{4 - 2\sqrt{3}} - \frac{\sqrt{3}+1}{\sqrt{3}-1}$$
   is an integer.

   Solution. $\sqrt{4-2\sqrt{3}} = \sqrt{3^2 - 2\sqrt{3} + \sqrt{1}^2} = \sqrt{(\sqrt{3}-1)^2} = |\sqrt{3}-1| = \sqrt{3}-1$. If multiply both numerator and the denominator of $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ by $\sqrt{3}+1$ we get:
$$\frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{\sqrt{3}^2 - 1^2} = \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}.$$
   Thus $A = -1 - 2 = -3$, which is an integer.

3. An isolated island has the shape of a circle. Initially there are 9 flowers on the circumference of the island: 5 of the flowers are red and the other 4 are yellow. During the summer 9 new flowers grow on the circumference of the island according to the following rule: between 2 old flowers of the same color a new red flower will grow, between 2 old flowers of different colors, a new yellow flower will grow. During the winter, the old flowers die, and the new survive. The same phenomenon repeats every year.

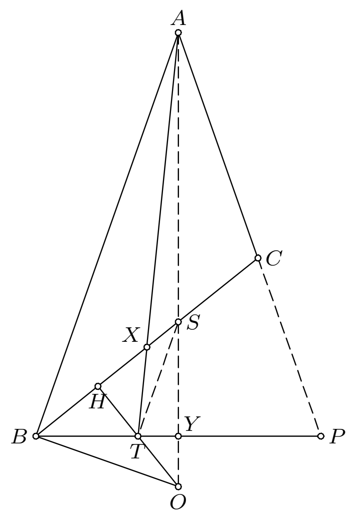
   Is it possible (for some configuration of initial 9 flowers) to get all red flowers after finitely many years?

   Solution. The answer is ”no”. Assume that we got all red flowers in the year n for the first time. Then in the year $n-1$ all the flowers were yellow. We will prove that this is impossible.

   Let's change the weird story into the one with the flowers labeled by 1 (instead of red) and 1 (instead of yellow). What really happens is that between two flowers a and b, the new flower will grow and will be labeled by ab. Notice that the initial product of all numbers is 1, and at the end of each winter the product of the numbers is 1 again, so it will never be equal to -1 hence it is impossible to get the configuration where all the flowers are yellow. This is a contradiction.

4. Given a triangle ABC, a circle k is tangent to the lines AB and AC at B and P. Let H be the foot of perpendicular from the center O of k to BC, and let T be the intersection point of OH and BP. Prove that AT bisects the segment BC.

   Solution. Let X be the intersection of AT and BC, S the intersection of BC and AO, and Y the intersection of BP and AO. Since $BY \perp SO$ and $OH \perp BS$, T is the orthocenter of $\triangle BOS$ and $ST \perp BO$, hence $ST \| AB$. Thus
$$ST : AB = TX : XA. \tag{1}$$
   However since $AB = AP$ and the triangles SYT and AYP are similar we get
$$ST : AP = TY : YP. \tag{2}$$
   From (1) and (2) we now get $XY \| AP$ and since Y is the midpoint of BP, XY is the middle line of $\triangle CBP$ and $BX = XC$.



5. A society has 100 members and every two members are either friends or enemies. Prove that there are two persons from the society that have an even number of common enemies.

Solution. Assume that for each two members the set of common enemies has an odd number of elements. Then the set of their common friends has also an odd number of elements. Fix a person a. Let $F = \{f_1, \dots, f_k\}$ be the set of friends of the member a, and $E = \{e_1, \dots, e_{99-k}\}$ the set of its enemies. First we will show that k is even. Assume that f_1 has m_1 friends inside F, f_2 has m_2 friends inside F, etc. Then $m_1 + m_2 + \cdots + m_k$ is equal twice the number of all friendships inside F. However each of the numbers $m_1, \dots, m_k$ must be odd because m_i represents the number of common friends for a and f_i. Since the sum of k odd numbers ended up being even, we conclude that k is even.

Thus, every member has an even number of friends (the argument in the previous paragraph works for any member of the society, not just a). Now we will caclulate the parity of the number of friendships between members of E and members of F. We will do that in two different ways and get the contradiction.

For every i, $1 \leq i \leq 99 - k$, e_i has to have an odd number of friends among F, because e_i and a have an odd number of common freinds. The number of elements of E is odd hence the total number of freindships between E and F has to be odd.

On the other hand, b_i ($1 \leq i \leq k$) has an odd number of friends inside F, a is also a friend of b_i, and the total number of friends of b_i has to be even. Thus, b_i has an even number of friends inside E. Hence the number of friendships between F and E is even. A contradiction!