Berkeley Math Circle

Monthly Contest 5
Due February 6, 2007

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 5
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

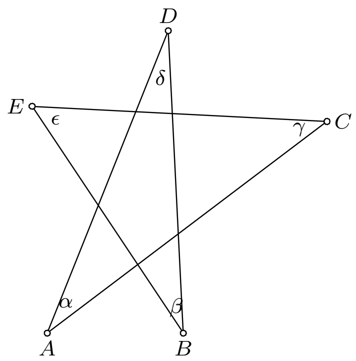
Enjoy solving these problems and good luck!

Problems

1. Do there exist 100 consecutive positive integers such that their sum is a prime number?

   Hint. What is the method for summing 100 consecutive numbers?

2. Let $ABCDE$ be a convex pentagon. If $\alpha = \angle DAC$, $\beta = \angle EBD$, $\gamma = ACE$, $\delta = \angle BDA$, and $\epsilon = \angle BEC$, as shown in the picture, calculate the sum $\alpha + \beta + \gamma + \delta + \epsilon$.



3. Bart has 17 and 19 dollar bills only.

   (a) Prove that these bills are fake.

   (b) Prove that there exists $m > 0$ such that for each $n \geq m$ Bart can give exactly n dollars to Lisa using his bills.

   Remark. Part (a) is worth 0 points but we would like to see your "proof".

4. Does there exist a convex polygon that can be partitioned into non-convex quadrilaterals?

5. The numbers from the table

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}$$

   satisfy the inequality

$$\sum_{i=1}^{n} |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n| \leq M,$$

   for every choice $x_i = \pm 1$. Prove that $|a_{11}| + |a_{22}| + \cdots + |a_{nn}| \leq M$.