Berkeley Math Circle
Monthly Contest 3 – Solutions

1. Is the number $\left|2^{3000} - 3^{2006}\right|$ bigger or smaller than $\frac{1}{2}$?

   Solution. Notice that $\left|2^{3000} - 3^{2006}\right|$ is a non-negative integer, so it is either 0 or bigger than $1/2$. However, this number is not 0 since $2^{3000} \neq 3^{2006}$, hence $\left|2^{3000} - 3^{2006}\right| > \frac{1}{2}$.

2. If a, b, c, d are positive real numbers such that $\frac{5a+b}{5c+d} = \frac{6a+b}{6c+d}$ and $\frac{7a+b}{7c+d} = 9$, calculate $\frac{9a+b}{9c+d}$.

   Solution. Let $\frac{5a+b}{5c+d} = \frac{6a+b}{6c+d} = k$. Then $5a + b = k(5c + d)$ and $6a + b = k(6c + d)$. Subtracting these two equations gives $a = kc$. Now we can easily get that $b = kd$. From $\frac{7a+b}{7c+d} = \frac{7kc+kd}{7c+d} = k = 9$ we get $\frac{9a+b}{9c+d} = 9$ as well.

3. Let $n > 3$ be an integer which is not divisible by 3. Two players A and B play the following game with $n \times n$ chocolate table. First, player A has to chose and remove one piece of the chocolate, without breaking other pieces. After his move, player B tries to partition the remaining chocolate into 3×1 (and 1×3) rectangles. If B manages to do so, then he/she is the winner. Otherwise the winner is A. Determine which player has a winning strategy and describe the strategy.

   Solution. The player A has a winning strategy. Imagine that the chocolate is painted in 3 colors, R(ed), G(reen), and B(lue) as shown in the table

$$\begin{array}{cccccccc}
R & G & B & R & G & B & R & \cdots \\
B & R & G & B & R & G & B & \cdots \\
G & B & R & G & B & R & G & \cdots \\
R & G & B & R & G & B & R & \cdots \\
B & R & G & B & R & G & B & \cdots \\
G & B & R & G & B & R & G & \cdots \\
R & G & B & R & G & B & R & \cdots \\
 & \vdots & & & \vdots & & & \vdots
\end{array}$$

   Notice that every 1×3 rectangle will contain exactly one red, one green, and one blue piece. Moreover the orignal table had equal number of green and blue pieces, and one red piece extra. If A takes out any green or blue piece, player B won't be able to fullfil the requirement, and A is the winner.

4. Given a triangle ABC, let D be the point of the ray BA such that $BD = BA + AC$. If K and M are points on the sides BA and BC, respectively, such that the triangles BDM and BCK have the same areas, prove that $\angle BKM = \frac{1}{2}\angle BAC$.

   Solution. Since the area of the triangle BDM is equal to $BD \cdot BM \cdot \sin\angle DBM$ and similar formula holds for the area of $\triangle BCK$ we immediately have that $BD \cdot BM = BC \cdot BK$. This implies that $\frac{BD}{BK} = \frac{BC}{BM}$ hence $KM \| CD$. Thus $\angle BKM = \angle BDC = \angle ADC$. The last angle is equal to $\frac{1}{2}\angle BAC$ since $\triangle DAC$ is isosceles.

5. Determine the greatest real number a such that the inequality

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \geq a(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5)$$

   holds for every five real numbers x_1, x_2, x_3, x_4, x_5.

   Solution. Note that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = \left(x_1^2 + \frac{x_2^2}{3}\right) + \left(\frac{2x_2^2}{3} + \frac{x_3^2}{2}\right) + \left(\frac{x_3^2}{2} + \frac{2x_4^2}{3}\right) + \left(\frac{x_4^2}{3} + x_5^2\right).$$

   Now applying the inequality $a^2 + b^2 \geq 2ab$ we get

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \geq \frac{2}{\sqrt{3}}(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5).$$

   This proves that $a \geq \frac{2}{\sqrt{3}}$. In order to prove $a \leq \frac{2}{\sqrt{3}}$ it is enough to notice that for $(x_1, x_2, x_3, x_4, x_5) = (1, \sqrt{3}, 2, \sqrt{3}, 1)$ we have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = \frac{2}{\sqrt{3}}(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5).$$