Berkeley Math Circle
Monthly Contest 3
Due December 5, 2006

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 3
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Is the number $\left|2^{3000} - 3^{2006}\right|$ bigger or smaller than $\frac{1}{2}$?
2. If a, b, c, d are positive real numbers such that $\frac{5a+b}{5c+d} = \frac{6a+b}{6c+d}$ and $\frac{7a+b}{7c+d} = 9$, calculate $\frac{9a+b}{9c+d}$.
3. Let $n > 3$ be an integer which is not divisible by 3. Two players A and B play the following game with $n \times n$ chocolate table. First, player A has to chose and remove one piece of the chocolate, without breaking other pieces. After that player B tries to partition the remaining chocolate into 3×1 (and 1×3) rectangles. If B manages to do so, then he/she is the winner. Otherwise the winner is A. Determine which player has a winning strategy and describe the strategy.
4. Given a triangle ABC, let D be the point of the ray BA such that $BD = BA + AC$. If K and M are points on the sides BA and BC, respectively, such that the triangles BDM and BCK have the same areas, prove that $\angle BKM = \frac{1}{2}\angle BAC$.
5. Determine the greatest real number a such that the inequality
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \geq a(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5)$$
holds for every five real numbers x_1, x_2, x_3, x_4, x_5.