Berkeley Math Circle

Monthly Contest 2
Due November 7, 2006

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 2
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Find all positive prime numbers p and q such that $p^2 - q^3 = 1$.

   Remark. p is prime if it has only two divisors: 1 and itself. The numbers $2, 3, 5, 7, 11, 13$ are prime, but $1, 4, 6, 8, 9$ are not.

   Hint: 2 is the oddest prime number because it's even. The others are just odd.

2. A line l and two points A and B are given in a plane in such a way that A belongs to l but B doesn't. Construct the circle k that passes through B and touches l at the point A.

   Note: Construction means that you have to explain how to do that using ruler and compas. Actual performance of the construction is not enough. You should do it, but the explanation is what is worth most of the credit.

3. If the sum of digits in a decimal representation of a natural number n is equal to 2006, prove that n can't be a perfect square of an integer.

4. Let $\triangle ABC$ be a triangle such that $\angle A = 90°$. Determine whether it is possible to partition $\triangle ABC$ into 2006 smaller triangles in such a way that

   1° Each triangle in the partition is similar to $\triangle ABC$;

   2° No two triangles in the partition have the same area.

   Explain your answer!

5. Let $S > 0$. If a, b, c, x, y, z are positive real numbers such that $a + x = b + y = c + z = S$, prove that

$$ay + bz + cx < S^2.$$