Berkeley Math Circle
Monthly Contest 1 – Solutions

1. If n is an integer, prove that the number
$$1 + n + n^2 + n^3 + n^4$$
is not divisible by 4.

   Solution. We will prove that $1 + n + n^2 + n^3 + n^4$ is odd number and the required claim will follow immediately since no odd number is divisible by 4. There are two possibilities for n– it can be even or odd. In the case when n is even, then $n + n^2 + n^3 + n^4$ is even as a sum of even numbers. Thus $1 + n + n^2 + n^3 + n^4$ is odd number. On the other hand, if n is odd then, all five numbers 1, n, n^2, n^3, and n^4 are odd and hence their sum must be also odd.

2. In how many different ways one can place 3 rooks on the cells of 6×2006 chessboard such that they don't attack each other?

   Solution. In order not to attack each other, the rooks no to rooks can share a row or a column. Since there are 6 columns in total, we have 20 possibilities for choosing 3 columns in which we will place our rooks. After choosing the columns, we will place our rooks one by one in these columns. The first rook can be placed in 2006 different ways. The second one, however can't be placed in the row that contains the first rook. Hence there are 2005 possibilities for the second rook. Similarly, the third rook can be placed in 2004 different ways. In total, there are $20 \cdot 2006 \cdot 2005 \cdot 2004$ ways to place 3 rooks in 6×2006 chessboard so that they don't attack each other.

3. Given a square table $n \times n$, two players A and B are playing the following game: At the beginning all cells of the table are empty, and the players alternate playing with coins. Player A has the first move, and in each of the moves a player will put a coin on some of the cells that doesn't contain a coin and is not adjacent to any of the cells that already contains a coin. The player who makes the last move is the winner. Which player has a winning strategy and what is the strategy?

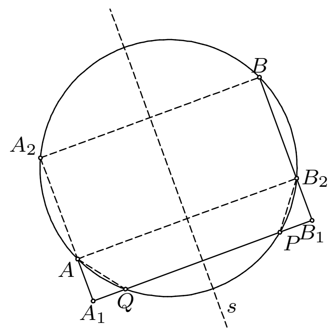
   Remark. The cells are adjacent if they share an edge.

   Solution. If n is even, B has the winning strategy – he just needs to put the coin in the centrally symmetric cell to the cell where A has put the coin in his prevoius move. Following this strategy, B will always have a free cell to put his coin, and thus he can never loose. Since the game has to end, A is the looser.

   If n is odd, A has a winning strategy: first he needs to put a coin in the central cell, and after that he puts the coin in the centrally symmetric cell to the cell where B has put his coin.

4. Given a circle k, let AB be its diameter. An arbitrary line l intersects the circle k at the points P and Q. If A_1 and B_1 are the feet of perpendiculars from A and B to PQ prove that $A_1P = B_1Q$.

   Solution. Let A_2 and B_2 be the intersections of the lines AA_1 and BB_1 with the circle k. Furthermore, let s be the line through the center of k perpendicular to l. Then AB_2BA_2 is a rectangle (because $AA_2 \| BB_2$ and $\angle AB_2B = 90^\circ$ since AB is a diameter) and s is an axis of symmetry of that rectangle. $A_1B_1B_2A$ is also a rectangle and s is also an axis of symmetry for that rectangle. Since k is also symmetric with repsect to s, we conclude that P and Q are symmetric with respect to s. Thus $A_1Q = B_1P$.



5. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(x) + y + 1) = x + f(y) + 1$ for every two integers x and y.

   Solution. Substituting $x = 0$ we get $f(f(0) + y + 1) = f(y) + 1$. Now we have $x + f(y) + 1 = f(f(x) + y + 1) = f[(f(x) + 1) + y] = f[f(f(0) + x + 1) + (y - 1) + 1] = f(y - 1) + (f(0) + x + 1) + 1$ from where we conclude that
$$f(y) = f(y - 1) + f(0) + 1. \qquad (1)$$
   Using the induction we obtain $f(n) = f(0) + n(f(0) + 1)$ for all integers n. If we set $y = 0$ in (1) we get $f(-1) = -1$ and substituting $y = -1$ in the original equation yields $f(f(x)) = x$. If we apply f to both sides of $f(n) = f(0) + n(f(0) + 1)$ we get $n = f(0) + [f(0) + n(f(0) + 1)](f(0) + 1)$ which is equivalent to $n = f(0)(f(0) + 2) + n(f(0) + 1)^2$ and this can be satisfied for all integers n if and only if $f(0) = 0$ or $f(0) = -2$. In the first case we get $f(n) = n$ and in the second $f(n) = -n - 2$. It is easy to verify that both functions satisfy the required conditions.