Berkeley Math Circle
Monthly Contest 1
Due October 10, 2006

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 1
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. If n is an integer, prove that the number
$$1 + n + n^2 + n^3 + n^4$$
is not divisible by 4.

   Hint: See what happens when you plug in small n's.

2. In how many different ways one can place 3 rooks on the cells of 6×2006 chessboard such that they don't attack each other?

   Hint: See what happens with the smaller boards first.

3. Given a square table $n \times n$, two players A and B are playing the following game: At the beginning all cells of the table are empty, and the players alternate playing with coins. Player A has the first move, and in each of the moves a player will put a coin on some of the cells that doesn't contain a coin and is not adjacent to any of the cells that already contains a coin. The player who makes the last move is the winner. Which player has a winning strategy and what is the strategy?

   Remark. The cells are adjacent if they share an edge.

4. Given a circle k, let AB be its diameter. An arbitrary line l intersects the circle k at the points P and Q. If A_1 and B_1 are the feet of perpendiculars from A and B to PQ prove that $A_1P = B_1Q$.

5. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(x) + y + 1) = x + f(y) + 1$ for every two integers x and y.