Berkeley Math Circle
Monthly Contest 7 – Solutions

1. If k is an integer, prove that the number $k^2 + k + 1$ is not divisible by 2006.

   Solution. The number $k^2 + k + 1 = k(k+1) + 1$ is odd, because $k(k+1)$ is even. Hence it can't be divisible by 2006.

2. Given an $n \times n$ matrix whose entries a_{ij} satisfy $a_{ij} = \dfrac{1}{i+j-1}$, n numbers are chosen from the matrix no two of which are from the same row or the same column. Prove that the sum of these n numbers is at least 1.

   Solution. Suppose that a_{ij} and a_{kl} are among the chosen numbers and suppose that $i < k$ and $j < l$. It is straightforward to show that $a_{ij} + a_{kl} \geqslant a_{il} + a_{kj}$. Hence, whenever a_{ij} and a_{kl} with $i < k$ and $j < l$ are among chosen numbers, we can lower the sum by replacing these two numbers with a_{il} and a_{kj}. Hence the smallest possible sum is when we choose $a_{1n}, a_{2,n-1}, \dots, a_{n1}$ – and in that case the sum is 1.

3. Given a triangle ABC such that $\angle B = 90^\circ$, denote by k the circle with center on BC that is tangent to AC. Denote by T a point of tangency of k and the tangent from A to k (different from AC). If B' is the midpoint of AC and M the intersection of BB' and AT, prove that $MB = MT$.

   Solution. Let O be the center of k. The quadrilateral $ABTO$ can be inscribed in a circle hence $\angle CBB' = \angle B'CB$ and $\angle TAO = \angle OAC$. Thus $\angle MTB = \angle ATB = \angle AOB = \angle ACO + \angle OAC = \angle B'BC + \angle TAO = \angle B'BC + \angle TBO = \angle TBM$ which immediately implies the given statement.

4. Let A be the number of 4-tuples (x, y, z, t) of positive integers smaller than 2006^{2006} such that

   $$x^3 + y^2 = z^3 + t^2 + 1,$$

   and let B the number of 4-tuples (x, y, z, t) of positive integers smaller than 2006^{2006} such that

   $$x^3 + y^2 = z^3 + t^2.$$

   Prove that $B > A$.

   Solution. For each natural number k, denote by l_k the number of pairs (x, y) such that $x^3 + y^2 = k$. Then $B = l_2^2 + l_3^2 + \cdots + l_r^2$ where r is the maximal k for which $l_k \neq 0$. Similarly, $A = l_2 l_3 + l_3 l_4 + \cdots + l_{r-1} l_r$. Now

   $$B = \frac{l_2^2}{2} + \frac{1}{2}(l_2^2 + l_3^2) + \frac{1}{2}(l_3^2 + l_4^2) + \cdots + \frac{1}{2}(l_{r-1}^2 + l_r^2) + \frac{l_r^2}{2} > l_2 l_3 + l_3 l_4 + \cdots + l_{r-1} l_r.$$

   We have used that $a^2 + b^2 \geq 2ab$.

5. Prove that the functional equations

   $$\begin{aligned} f(x+y) &= f(x) + f(y), \\ \text{and} \quad f(x+y+xy) &= f(x) + f(y) + f(xy) \quad (x, y \in \mathbb{R}) \end{aligned}$$

   are equivalent.

   Solution. Let us assume that $f(x+y) = f(x) + f(y)$ for all reals. In this case we trivially apply the equation to get $f(x+y+xy) = f(x+y) + f(xy) = f(x) + f(y) + f(xy)$. Hence the equivalence is proved in the first direction.

   Now let us assume that $f(x+y+xy) = f(x) + f(y) + f(xy)$ for all reals. Plugging in $x = y = 0$ we get $f(0) = 0$. Plugging in $y = -1$ we get $f(x) = -f(-x)$. Plugging in $y = 1$ we get $f(2x+1) = 2f(x) + f(1)$ and hence $f(2(u+v+uv)+1) = 2f(u+v+uv) + f(1) = 2f(uv) + 2f(u) + 2f(v) + f(1)$ for all real u and v. On the other hand, plugging in $x = u$ and $y = 2v+1$ we get $f(2(u+v+uv)+1) = f(u + (2v+1) + u(2v+1)) = f(u) + 2f(v) + f(1) + f(2uv+u)$. Hence it follows that $2f(uv) + 2f(u) + 2f(v) + f(1) = f(u) + 2f(v) + f(1) + f(2uv+u)$, i.e.,

   $$f(2uv+u) = 2f(uv) + f(u). \tag{1}$$

   Plugging in $v = -1/2$ we get $0 = 2f(-u/2) + f(u) = -2f(u/2) + f(u)$. Hence, $f(u) = 2f(u/2)$ and consequently $f(2x) = 2f(x)$ for all reals. Now (1) reduces to $f(2uv+u) = f(2uv) + f(u)$. Plugging in $u = y$ and $x = 2uv$, we obtain $f(x) + f(y) = f(x+y)$ for all nonzero reals x and y. Since $f(0) = 0$, it trivially holds that $f(x+y) = f(x) + f(y)$ when one of x and y is 0.