Berkeley Math Circle
Monthly Contest 4 – Solutions

1. A cube $3 \times 3 \times 3$ is made of cheese and consists of 27 small cubical cheese pieces arranged in the $3 \times 3 \times 3$ pattern. A mouse is eating the cheese in such a way that it starts at one of the corners and eats smaller pieces one by one. After he finishes one piece, he moves to the adjacent piece (pieces are adjacent if they share a face). Is it possible that the last piece mouse has eaten is the central one?

   Remark: Pieces don't fall down if a piece underneath is eaten first.

   Solution. Color the pieces of chiese alternatively in red and green such that corners are green and any two adjacent cubes are of different colors. We easily see that the mouse is moving always from the cube of one color to the cube of the other color. There are 14 green and 13 red cubes, the central cube being red. Since mouse has started from green piece, it will finish at the green piece (after 27 moves), hence it can't finish at the central cube.

2. Let M be the midpoint of the side AC of triangle ABC. If N is the point on the side AB, O intersection of the lines BM and CN, and if the areas of triangles BON and COM are equal, prove that N is the midpoint of AB.

   Solution. Since the areas of $\triangle BON$ and $\triangle COM$ are equal we see that the areas of triangles $\triangle BCN$ and $\triangle CBM$ are also equal. Since these two triangles share the side, they must have the corresponding altitudes equal. Hence the length of perpendiculars from M and N to BC are equal, implying that $NM \| BC$. Thus MN is the midsegment of $\triangle ABC$ and consequently N is the midpoint of AB.

3. Determine whether the number
$$\frac{1}{2\sqrt{1}+1\sqrt{2}}+\frac{1}{3\sqrt{2}+2\sqrt{3}}+\frac{1}{4\sqrt{3}+3\sqrt{4}}+\cdots+\frac{1}{100\sqrt{99}+99\sqrt{100}}$$
   is rational or irrational. Explain your answer.

   Solution. Notice that
$$\begin{aligned}\frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}} &= \frac{1}{\sqrt{n(n+1)}}\cdot\frac{1}{\sqrt{n+1}+\sqrt{n}}\\ &= \frac{1}{\sqrt{n(n+1)}}\cdot\frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}\\ &= \frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}.\end{aligned}$$
   Now we see that the given sum is equal to $1-\frac{1}{\sqrt{100}}=\frac{9}{10}$.

4. Let $a_1, a_2, \ldots, a_n$ be positive real numbers whose sum is equal to 1. If
$$\begin{aligned} S = & \frac{a_1^2}{2a_1}+\frac{a_1a_2}{a_1+a_2}+\frac{a_1a_3}{a_1+a_3}+\cdots+\frac{a_1a_n}{a_1+a_n}\\ & +\frac{a_2a_1}{a_2+a_1}+\frac{a_2^2}{2a_2}+\frac{a_2a_3}{a_2+a_3}+\cdots+\frac{a_2a_n}{a_2+a_n}\\ & \vdots\\ & +\frac{a_na_1}{a_n+a_1}+\frac{a_na_2}{a_n+a_2}+\frac{a_na_3}{a_n+a_3}+\cdots+\frac{a_n^2}{2a_n},\end{aligned}$$
   prove that $S \le \frac{n}{2}$.

   Solution. It is easy to show that $\left(\frac{a+b}{2}\right)^2 \ge ab$. Now applying this inequality to a_1, a_2 we get $\frac{a_1a_2}{a_1+a_2} \le \frac{[(a_1+a_2)/2]^2}{a_1+a_2} = \frac{a_1+a_2}{4}$. We get similar inequalities for all terms of the given sum. Now, using that $a_1+a_2+a_3+\cdots+a_n=1$ we get:
$$\frac{a_1^2}{2a_1}+\frac{a_1a_2}{a_1+a_2}+\frac{a_1a_3}{a_1+a_3}+\cdots+\frac{a_1a_n}{a_1+a_n} \le \frac{a_1+a_1}{4}+\frac{a_1+a_2}{4}+\frac{a_1+a_3}{4}+\cdots+\frac{a_1+a_n}{4}=\frac{na_1+1}{4}.$$

Similarly

$$\frac{a_2a_1}{a_2+a_1}+\frac{a_2^2}{2a_2}+\frac{a_2a_3}{a_2+a_3}+\cdots+\frac{a_2a_n}{a_2+a_n}\leq\frac{na_2+1}{4},\ \text{etc.}$$

Adding these inequalities and using again that $a_1+\cdots+a_n=1$ we get the desired inequality. Equality holds if and only if $a_1=a_2=\cdots=a_n=1/n$.

5. Each of three schools contain n students. Each student has at least $n+1$ friends among students of the other two schools. Prove that there are three students, all from different schools who are friends to each other. (Friendship is symmetric: If A is a friend to B, then B is a friend to A.)

   Solution. Suppose to the contrary that there doesn't exist three students from different schools who know each other. Denote schools by A, B and C. Consider the student (or one of the students if there are more of them) who has the maximal number of friends in one of the schools. Let a be the name of that student, suppose he is from the school A and suppose that he has k friends denoted by $b_1, b_2, \ldots, b_k$ from the school B. Then he must have at least $n+1-k$ friends in the school C, say $c_1, c_2, \ldots, c_{n+1-k}$. Then student c_1 doesn't know any of $b_1, b_2, \ldots, b_k$ (otherwise, if he knows b_i, then a, b_i and c_1 would be the desired triple), so he can have at most $n-k$ friends in the school B. Since c_1 has at least $n+1$ friends in the schools B and A, he must have at least $n+1-(n-k)=k+1$ friends in A. However, this is impossible since we assumed that k is the maximal number of friends a student can have in one of the schools.

   The obtained contradiction proves the result.