Berkeley Math Circle
Monthly Contest 4
Due January 17, 2006

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 4
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A cube $3 \times 3 \times 3$ is made of cheese and consists of 27 small cubical cheese pieces arranged in the $3 \times 3 \times 3$ pattern. A mouse is eating the cheese in such a way that it starts at one of the corners and eats the small pieces one by one. After the mouse finishes one piece, it moves to an adjacent piece (pieces are adjacent if they share a face). Is it possible that the last piece mouse eats is the central one?

   Remark: Pieces don't fall down if a piece underneath is eaten first.

2. Let M be the midpoint of the side AC of triangle ABC. If N is the point on the side AB, O intersection of the lines BM and CN, and if the areas of triangles $\triangle BON$ and $\triangle COM$ are equal, prove that N is the midpoint of AB.

3. Determine whether the number
$$\frac{1}{2\sqrt{1}+1\sqrt{2}}+\frac{1}{3\sqrt{2}+2\sqrt{3}}+\frac{1}{4\sqrt{3}+3\sqrt{4}}+\cdots+\frac{1}{100\sqrt{99}+99\sqrt{100}}$$
is rational or irrational. Explain your answer.

4. Let $a_1, a_2, \ldots, a_n$ be positive real numbers whose sum is equal to 1. If
$$\begin{aligned} S \quad = \quad & \frac{a_1^2}{2a_1}+\frac{a_1a_2}{a_1+a_2}+\frac{a_1a_3}{a_1+a_3}+\cdots+\frac{a_1a_n}{a_1+a_n} \\ & +\frac{a_2a_1}{a_2+a_1}+\frac{a_2^2}{2a_2}+\frac{a_2a_3}{a_2+a_3}+\cdots+\frac{a_2a_n}{a_2+a_n} \\ & \vdots \\ & +\frac{a_na_1}{a_n+a_1}+\frac{a_na_2}{a_n+a_2}+\frac{a_na_3}{a_n+a_3}+\cdots+\frac{a_n^2}{2a_n}, \end{aligned}$$
prove that $S \leq \frac{n}{2}$.

   Remark: S can be written in the sumation notation as $S = \sum_{i,j=1}^{n} \frac{a_ia_j}{a_i+a_j}$.

5. Each of three schools contain n students. Each student has at least $n+1$ friends among students of the other two schools. Prove that there are three students, all from different schools who are friends to each other. (Friendship is symmetric: If A is a friend to B, then B is a friend to A.)