Berkeley Math Circle
Monthly Contest 3 – Solutions

1. Given n positive real numbers $a_1, a_2, \ldots, a_n$ such that $a_1 a_2 \cdots a_n = 1$, prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n.$$

   When does the equality hold?

   Solution. By the inequality $a + b \geq 2\sqrt{ab}$ which holds for positive numbers a, b (and equality is if and only if $a = b$), we see that $1 + a_1 \geq 2\sqrt{a_1}$, $1 + a_2 \geq 2\sqrt{a_2}$, $\ldots$, $1 + a_n \geq 2\sqrt{a_n}$. Multiplying these inequalities we get $(1 + a_1) \cdots (1 + a_n) \geq 2^n \sqrt{a_1 \cdots a_n} = 2^n$. The equality holds if and only if $a_1 = a_2 = \cdots = a_n = 1$.

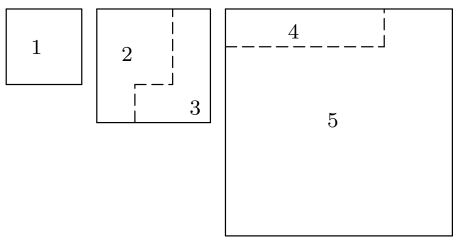
2. There are three prisoners in a prison. A warden has 2 red and 3 green hats and he has decided to play the following game: He puts the prisoners in a row one behind the other and on the had of each prisoner he puts a hat. The first prisoner in the row can't see any of the hats, the second prisoner can see only the hat at the had of the first one, and the third prisoner can see the hats of the first two prisoners. If some of the prisoners tells the color of his own hat, he is free; but if he is wrong, the warden will kill him. If a prisoner remain silent for sufficiently long, he is returned to his cell. Of course, each of them would like to be rescued from the prison, but if he isn't sure about the color of his hat, he won't guess.

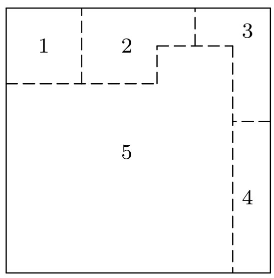
   After noticing that second and third prisoner are silent for a long time, first prisoner (the one who doesn't see any hat) has concluded the color of his hat and told that to the warden. What is the color of the hat of the first prisoner? Explain your answer! (All prisoners know that there are 2 red and 3 green hats in total and all of them are good at mathematics.)

   Solution. If the first two prisoners had red hats, the third one won't be silent (he would conclude that his hat is green). Hence, at least one of the first two prisoners has a green hat, and everybody knows that (because the third prisoner is silent). Thus if the first prisoner had red hat, the second one would conclude that he must be the one with the green hat. However, the second prisoner was silent and the hat at the hat of the first prisoner must be green.

3. Given three squares of dimensions 2×2, 3×3, and 6×6, choose two of them and cut each into 2 figures, such that it is possible to make another square from the obtained 5 figures.

   Solution. The solution is shown in the picture below





4. Let $ABCD$ be a rectangle. Let E be the foot of perpendicular from A to BD. Let F be an arbitrary point of the diagonal BD between D and E. Let G be the intersection of the line CF with the perpendicular from B to AF. Let H be the intersection of the line BC with the perpendicular from G to BD. Prove that $\sphericalangle EGB = \sphericalangle EHB$.

Solution. Let X be the intersection of lines BG and AE. Since $BX \perp AF$ and $AX \perp BF$, X is the orthocenter of $\triangle ABF$ hence $FX \perp AB \Rightarrow FX \parallel BC \parallel AD$. It follows:

$$\frac{EF}{ED} = \frac{FX}{DA} = \frac{FX}{BC} = \frac{GF}{GC} \ ,$$

(the first equality follows from $\triangle EFX \sim \triangle EDA$, and the last from $\triangle GFX \sim \triangle GCB$). We conclude that $GE \parallel CD \perp BH$. Since $BE \perp GH$ it follows that E is the orthocenter of $\triangle GHB$ and $\sphericalangle EGB = \sphericalangle EHB$ follows easily (if we denote by H' and G' the intersections of GE and HE, the triangles $GG'B$ and $HH'B$ are similar).

5. Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, where n is an integer?

   Solution. Suppose that there exist integers n and m such that $m^3 = 3n^2 + 3n + 7$. Then from $m^3 \equiv 1 \pmod 3$ it follows that $m = 3k + 1$ for some $k \in \mathbb{Z}$. Substituting into the initial equation we obtain $3k(3k^2 + 3k + 1) = n^2 + n + 2$. It is easy to check that $n^2 + n + 2$ cannot be divisible by 3, and so this equality cannot be true. Therefore our equation has no solutions in integers.