Berkeley Math Circle
Monthly Contest 3
Due December 20, 2005

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 3
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Given n positive real numbers $a_1, a_2, \dots, a_n$ such that $a_1 a_2 \cdots a_n = 1$, prove that

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n.$$

   When does the equality hold?

2. There are three prisoners in a prison. A warden has 2 red and 3 green hats and he has decided to play the following game: He puts the prisoners in a row one behind the other and on the had of each prisoner he puts a hat. The first prisoner in the row can't see any of the hats, the second prisoner can see only the hat at the had of the first one, and the third prisoner can see the hats of the first two prisoners. If some of the prisoners tells the color of his own hat, he is free; but if he is wrong, the warden will kill him. If a prisoner remain silent for sufficiently long, he is returned to his cell. Of course, each of them would like to be rescued from the prison, but if he isn't sure about the color of his hat, he won't guess.

   After noticing that second and third prisoner are silent for a long time, first prisoner (the one who doesn't see any hat) has concluded the color of his hat and told that to the warden. What is the color of the hat of the first prisoner? Explain your answer! (All prisoners know that there are 2 red and 3 green hats in total and all of them are good at mathematics.)

3. Given three squares of dimensions 2×2, 3×3, and 6×6, choose two of them and cut each into 2 figures, such that it is possible to make another square from the obtained 5 figures.

4. Let $ABCD$ be a rectangle. Let E be the foot of perpendicular from A to BD. Let F be an arbitrary point of the diagonal BD between D and E. Let G be the intersection of the line CF with the perpendicular from B to AF. Let H be the intersection of the line BC with the perpendicular from G to BD. Prove that $\sphericalangle EGB = \sphericalangle EHB$.

5. Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, where n is integer?

   Hint: Using mod "a suitable number" will be helpful.