Berkeley Math Circle
Monthly Contest 1 – Solutions

1. Whenever the child takes two apples from the tree, the number of green apples will decrease by 2 or remain unchanged – the parity of green apples remains the same. Since that number was odd at the beginning, it must be odd at the end. Hence, the last apple must be green.

2. It is easy to verify that x^3 has only three remainders upon division by 9, namely 0, 1 and 8. Writing down all possible combinations, we see that $x^3 + y^3 + z^3 \not\equiv 5 \pmod 9$, while $500 \equiv 5 \pmod 9$. This proves the statement.

3. F is the orthocenter of the triangle ABE (or E is the orthocenter of $\triangle ABF$) hence $EF \perp AB$. From $\sphericalangle CSD = 90°$ we conclude that $\sphericalangle CAD = 45°$ and $\triangle ACF$ is right-angled with $AC = CF$. Since $\sphericalangle ECF = \sphericalangle BCA = 90°$ and $\sphericalangle EFC = \sphericalangle BAC$ (both of them are equal to $90° - \sphericalangle CEF$, as can be easily seen from $\triangle CFE$ and triangle formed by lines AE, EF and AB), triangles ECF and BCA are congruent, hence $EF = AB$.

4. Recall that $x^2 + y^2 \geq 2xy$, where equality holds if and only if $x = y$. Applying this inequality to the pairs of numbers $(a/2, b)$, $(a/2, c)$, and $(a/2, d)$ yields:

$$\begin{aligned} \frac{a^2}{4} + b^2 &\geq ab, \\ \frac{a^2}{4} + c^2 &\geq ac, \\ \frac{a^2}{4} + d^2 &\geq ad. \end{aligned}$$

   Note also that $a^2/4 > 0$. Adding these four inequalities gives us $a^2 + b^2 + c^2 + d^2 \geq a(b + c + d)$. Equality can hold only if all the inequalities were equalities, i.e. $a^2 = 0$, $a/2 = b$, $a/2 = c$, $a/2 = d$. Hence $a = b = c = d = 0$ is the only solution of the given equation.

5. Yes, see the following example: