Berkeley Math Circle
Monthly Contest 1
Due November 1, 2005

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 1
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A magical tree contains 2005 green and 2006 red apples. Every time a child climbs the tree, he (she) eats 2 apples. After that a miracle happens: when the child takes 2 apples of the same color, one red apple grows on the tree; when the child takes 2 apples of different colors, one green apple grows on the tree.

   What will be the color of the last apple? Why?

   Hint: Try smaller size trees first to see what is happening.

2. Prove that no three integers x, y, z satisfy
$$x^3 + y^3 + z^3 = 500.$$

   Note. An *integer* is a whole positive or negative number, or 0, i.e. 0, +1, -1, +2, -2, etc.

   Hint: What are all possible remainders of numbers x^3 when divided by 9? A suitable section in the book "Mathematical Circles" will be hepful here.

3. Given a circle k with center S and diameter AB, let C and D be two points on the circle on same side of AB such that $\angle CSD = 90°$. The lines AC and BD intersect at E and lines AD and BC intersect at F. Prove that EF is perpendicular to AB and $EF = AB$.

   Hint: Do you see an orthocenter in a triangle? (An orthocenter is the intersection points of the altitudes of a triangle.) In addition, the concepts of inscribed and central angles in a circle are crucial here, as well as similar and congruent triangles.

4. Find all real numbers a, b, c and d such that
$$a^2 + b^2 + c^2 + d^2 = a(b + c + d).$$

   Hint: The famous inequality $x^2 + y^2 \geq 2xy$ is helpful here, however, you may have to apply it in a nontrivial way.

5. Is it possible to divide a square into 10 convex pentagons? If yes, provide such a division. If no, justify clearly your answer.

   Remark. A polygon is *convex* if all its internal angles are less than $180°$.