Berkeley Math Circle
Monthly Contest 7
Due April 11, 2006

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 7
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. If k is an integer, prove that the number $k^2 + k + 1$ is not divisible by 2006.

2. Given an $n \times n$ matrix whose entries a_{ij} satisfy $a_{ij} = \dfrac{1}{i+j-1}$, n numbers are chosen from the matrix no two of which are from the same row or the same column. Prove that the sum of these n numbers is at least 1.

3. Given a triangle ABC such that $\angle B = 90^\circ$, denote by k the circle with center on BC that is tangent to AC. Denote by T a point of tangency of k and the tangent from A to k (different from AC). If B' is the midpoint of AC and M the intersection of BB' and AT, prove that $MB = MT$.

4. Let A be the number of 4-tuples (x, y, z, t) of positive integers smaller than 2006^{2006} such that

$$x^3 + y^2 = z^3 + t^2 + 1,$$

and let B the number of 4-tuples (x, y, z, t) of positive integers smaller than 2006^{2006} such that

$$x^3 + y^2 = z^3 + t^2.$$

Prove that $B > A$.

5. Prove that the functional equations

$$\begin{aligned} f(x+y) &= f(x) + f(y), \\ \text{and} \qquad f(x+y+xy) &= f(x) + f(y) + f(xy) \quad (x, y \in \mathbb{R}) \end{aligned}$$

are equivalent.