Berkeley Math Circle
Monthly Contest 6 – Solutions

1. Find all pairs (m, n) of natural numbers such that $200m + 6n = 2006$.

   Solution. First we divide both sides of the equation by 2 and get: $100m + 3n = 1003$. Since m and n are natural numbers we immediately get that $m \leq 10$. Since $3n$ is divisible by 3 and 1003 gives remainder 1 upon division by 3, we conclude that $100m$ must also give the remainder 1 upon division by 3. Since $100m = 99m + m$ and $99m$ is divisible by 3 we see that m must give remainder 1 when divided by 3. Thus m has to be one of the numbers $1, 4, 7, 10$. Corresponding ns are $301, 201, 101$ and 1, respectively.

2. Circles k_1 and k_2 intersect at points A and B. Line l is the common tangent to these circles and it touches k_1 at C and k_2 at D such that B belongs to the interior of the tirangle ACD. Prove that $\angle CAD + \angle CBD = 180°$.

   Solution. Here we use the fact that the angle between the tangent and the chord is equal to the peripheral angle corresponding to the chord (this fact is easy to verify and can be found in any book on geometry). We have $\angle CAB = \angle DCB$ and $\angle DAB = \angle DCB$. Hence $\angle CAD = \angle CAB + \angle DAB = \angle DCB + \angle DCB = 180° - \angle DBC$. The statement now follows directly.

3. If x and y are two positive numbers less than 1, prove that

$$\frac{1}{1-x^2} + \frac{1}{1-y^2} \geq \frac{2}{1-xy}.$$

   Solution. First we use the inequality $a + b \geq 2\sqrt{ab}$ and get $\frac{1}{1-x^2} + \frac{1}{1-y^2} \geq \frac{2}{\sqrt{(1-x^2)(1-y^2)}}$. Now we notice that $(1-x^2)(1-y^2) = 1 + x^2y^2 - x^2 - y^2 \leq 1 + x^2y^2 - 2xy = (1-xy)^2$ which implies that $\frac{2}{\sqrt{(1-x^2)(1-y^2)}} \geq \frac{2}{1-xy}$ and this completes the proof.

4. If n is natural number such that $2n + 1$ and $3n + 1$ are perfect squares, prove that $5n + 1$ can't be a prime number.

   Solution. This problem contains a typo in its original formulation. Aaron Wilkowski, one of the BMC contestants, found a counter-example, e.g. $n = 3960$. Indeed, $2n + 1 = 89^2$, $3n + 1 = 109^2$, and $5n + 1 = 19801$ which is a prime.

   The correct formulation of the problem is: If n is natural number such that $2n + 1$ and $3n + 1$ are perfect squares, prove that $5n + 3$ can't be a prime number.

   The solution is:

   Suppose that $5n + 3$ is a prime number. Let x and y be natural numbers such that $x^2 = 2n + 1$ and $y^2 = 3n + 1$. Then $5n + 3 = 4(2n + 1) - (3n + 1) = 4x^2 - y^2 = (2x - y)(2x + y)$. Since $5n + 1$ is prime number and x and y are natural numbers we must have $2x - y = 1$ implying that $y = 2x - 1$. Since $n = y^2 - x^2$ we have that $x^2 = 2n+1 = 2(y^2 - x^2) + 1 = 2((2x-1)^2 - x^2) + 1 = 6x^2 - 8x + 3$. Thus $5x^2 - 8x + 3 = 0$ and the solutions are $x = 1$ or $x = 3/5$. Since $x^2 = 2n + 1$ for a natural number n neither of these x would work and we have a contradiction.

5. Every two members of a certain society are either *friends* or *enemies*. Suppose that there are n members of the society, that there are exactly q pairs of friends, and that in every set of three persons there are two who are enemies to each other. Prove that there is at least one member of the society among whose enemies there are at most $q \cdot \left(1 - \frac{4q}{n^2}\right)$ pairs of friends.

   Solution. Denote by S the set of all members of the society, by A the set of all pairs of friends, and by N the set of all pairs of enemies. For every $x \in S$, denote by $f(x)$ number of friends of x and by $F(x)$ number of pairs of friends among enemies of x. It is easy to prove:

$$q = |A| = \frac{1}{2}\sum_{x \in S} f(x);$$

$$\sum_{\{a,b\} \in A} (f(a) + f(b)) = \sum_{x \in S} f^2(x).$$

   If a and b are friends, then the number of their common enemies is equal to $(n-2) - (f(a)-1) - (f(b)-1) = n - f(a) - f(b)$. Thus

$$\frac{1}{n}\sum_{x \in S} F(x) = \frac{1}{n}\sum_{\{a,b\} \in A}(n - f(a) - f(b)) = q - \frac{1}{n}\sum_{x \in S} f^2(x).$$

Using the inequality between arithmetic and quadratic mean on the last expression, we get

$$\frac{1}{n}\sum_{x\in S} F(x) \leq q - \frac{4q^2}{n^2}$$

and the statement of the problem follows immediately.