Berkeley Math Circle
Monthly Contest 5
Due February 14, 2006

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 5
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. There are n birds sitting on the circumference of a circle. Some of the birds are owls and the others are eagles. One morning the eagles become angry and start a war: Every morning each eagle looks at its two neighbours, and if one (or both) of them is an owl it kills that neighbour (or both of them, if they are owls). Every night each owl looks at its neighbours and if one (or both) of them is an eagle, the owl kills it (or both neighbours, if they are eagles). This war lasts until only one sort of birds remain on the circle. Prove that for any $n > 1$ it is possible to have exactly 2 birds of the same kind left.

   Note: Suitable initial configurations of birds with an explanation of why they work is one possible way to solve the problem.

   Remark: In the morning every eagle will kill only its immediate neighbours (if they are owls), it won't start going around to kill more owls. If two eagles have the same neighbour which happen to be an owl, they will kill it together. The same is true for owls in the evening.

   Hint: Check for smaller numbers n first.

2. If $p_1, p_2, \dots, p_k$ are different positive prime numbers, prove that there is no positive integer n such that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = \frac{1}{n}.$$

   Hint: Clear denominators and use a divisibility argument.

3. Given n numbers $x_1, x_2, \dots, x_n$ each of which is equal to 1 or -1, suppose that $x_1x_2 + x_2x_3 + \cdots + x_nx_1 = 0$. Prove that n is divisible by 4.

4. The distance between the centers of the circles k_1 and k_2 with radii r is equal to r. Points A and B are on the circle k_1, symmetric with respect to the line connecting the centers of the circles. Point P is an arbitrary point on k_2. Prove that

$$PA^2 + PB^2 \geq 2r^2.$$

   When does equality hold?

   Hint: If A_1 is the midpoint of the side BC of $\triangle ABC$ then $AB^2 + AC^2 = 2(AA_1^2 + BC^2)$.

5. Initially, number 1 is 2006 times written on the blackboard. A student is playing the following game: at each step he replaces two of the numbers from the blackboard by the quarter of their sum. After 2005 steps only one number will remain at the blackboard. Prove that that number must be greater than or equal to $1/2006$.