Berkeley Math Circle
Monthly Contest 2 – Solutions

1. The frog can't manage the described trip. We will take advantage of the coloring of the chessboard. Notice that at each step frog is jumping to the square that is of different color than the square from which frog has jumped. Since the frog will make a total of 63 jumps, it must finish the trip on a cell that is of different color than the cell from which the trip has started. It remains to notice that the lower-left and the upper-right corners are of the same color.

2. Since AC is the diameter of k, we conclude that $\angle AEC = 90^\circ$. Since $AC \perp BC$, BC is a tangent to the circle k. Hence, DE and DC are two segments that are tangent to k which implies that $DE = DC$. We now have that D is the point on hypothenuse of $\triangle ECB$ such that $DE = DC$ which in turn gives that D is the midpoint of BC and $DE = DB$. The statement is proved.

3. If we apply the inequality $x^2 + y^2 \geq 2xy$ to the numbers $x = \dfrac{ab}{c}$ and $y = \dfrac{bc}{a}$ we get
$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} \geq 2b^2. \tag{1}$$
Similarly we get
$$\frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} \geq 2c^2, \text{ and} \tag{2}$$
$$\frac{c^2a^2}{b^2} + \frac{a^2b^2}{c^2} \geq 2a^2. \tag{3}$$
Summing up (1), (2) and (3) gives
$$2\left(\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}\right) \geq 2(a^2+b^2+c^2) = 2,$$
hence $S \geq 1$. The equality holds if and only if $\dfrac{ab}{c} = \dfrac{bc}{a} = \dfrac{ca}{b}$, i.e. $a = b = c = \dfrac{1}{\sqrt{3}}$.

4. Divide the numbers into 1002 pairs in the following way: $\{1, 2004\}$, $\{2, 2003\}$, $\{3, 2002\}$, $\ldots$, $\{1002, 1003\}$. Since $1002 + k$ numbers are blue, there are at least k pairs whose both elements are blue. If we choose $2k$ numbers from these pairs their sum will be $2005k$ (since the sum of numbers in each pair is equal to 2005). The number $2005k$ is, obviously, divisible by 2005.

5. Suppose that there are no four colinear red points with the stated property. Let P be any blue point (there must be at least one blue point). Consider the circle k with center P and radius 1. If some point on the circle is blue, then the claim will follow immediately. Thus suppose that all points on the circle are red. Let $ABCDEF$ be an arbitrary regular hexagon whose vertices belong to k. Let X be intersection of AB with CD, Y the intersection of CD with EF, and Z the intersection point of EF with AB. If at least two of the points X, Y, Z are red (say X and Y) then we can find four colinear red points (in our case, X, C, D, Y) with the stated property, and that would be a contradiction. Hence at least two of the points X, Y, Z are blue.

   Now, consider the rotation with the center P by which the points A, B, C, D, E, F, X, Y, Z are mapped to the points A', B', C', D', E', F', X', Y', Z' such that $XX' = YY' = ZZ' = 1$. By the same argument we get that two of the points X', Y', Z' are blue. The problem is now solved, since at least one of the segments XX', YY', ZZ' has both ends painted in blue.