Berkeley Math Circle
Monthly Contest 3
Solutions

1. Find all integer solutions to $xy = 2003(x + y)$

We get

$$2003^2 - 2003x - 2003y - xy = 2003^2$$

$$(2003 - x)(2003 - y) = 2003^2$$

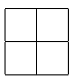
So $(2003 - x)$ is a divisor of 2003^2. As 2003 is a prime number, we conclude that $2003 - x$ is $-2003^2, -2003, -1, 1, 2003$, or 2003^2, and x. Then for x, y we obtain

$$\begin{array}{lllllll} x & = & 2003^2 + 2003 = 4014012, & y & = & 2004, \\ x & = & 2003 + 2003 = 4006, & y & = & 4006, \\ x & = & 1 + 2003 = 2004, & y & = & 4014012, \\ x & = & -1 + 2003 = 2002, & y & = & -401006, \\ x & = & -2003 + 2003 = 0, & y & = & 0, \\ x & = & -2003^2 + 2002 = -401006, & y & = & 2002. \end{array}$$

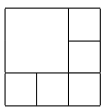
2. Show that for each $n \geq 17$ one can cut a square into n smaller squares.

One can cut a square into n smaller squares for all $n \geq 6$. Here is how.

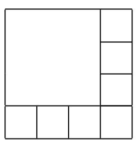
For $n = 4$



For $n = 6$



For $n = 8$



Suppose we know how to cut a square into n smaller squares. If we take one of the square "pieces" and cut it onto four smaller squares (as in the picture above for $n = 4$. Now we have cut our original square into $n+3$ pieces. Since we know how to cut a square into 4 pieces, we can cut it into $7, 10, \ldots, 3k+1, \ldots$ pieces. Similarly, starting with 6 pieces we can get $6, 9, \ldots, 3k, \ldots$ pieces, and from 8 pieces - $8, 11, \ldots, 3k+2, \ldots$ pieces. Hence we can get any number of pieces bigger or equal to 6.

3. In each cell of $n \times n$ table there is an arrow pointing in one of eight principal directions (i.e. one of the following: $\rightarrow, \leftarrow, \uparrow, \downarrow, \nearrow, \searrow, \nwarrow, \swarrow$), in such a way that the arrows in any two adjacent cells form an angle of no more than 45 degrees, and the arrows in any two cells adjacent diagonally (i.e. sharing a vertex) form an angle of no more than 90 degrees. We start at some cell and follow the arrows (e.g. if the arrow in our cell is $\uparrow$ we move up by one, if it is $\searrow$ we move diagonally down and to the right, etc.). Prove that we will eventually escape from the table (i.e. will reach a cell on the boundary of the table where the arrow will point "out of the table").

Suppose that there exists such a table of arrows and a cell in it so that we can follow the arrows without ever escaping the table. Then, since the number of cells in the table is finite, we must at some point return to a cell we have already visited, and then continue in a loop. Let's rotate all the arrows in our original $n \times n$ table by 45 degrees, so that the arrows in the cells along our loop point **into** the loop after the rotation. Note that because the adjacent arrows of the loop are at no more than 90 degrees, the rotated arrows will indeed point to the cells inside of the loop. The new table of (rotated) arrows still has the property that any two adjacent cells form an angle of no more than 45 degrees, and the arrows in any two cells adjacent diagonally form an angle of no more than 90 degrees. If we now start at some point on the loop and follow the (rotated) arrows, then since all the arrows on the boundary of the loop point inward, we will never escape from inside of the loop, and so there is going to be a yet smaller loop inside of our first loop. We can now repeat the process, obtaining smaller and smaller loops along the way. But the number of cells inside the loop (counting those on the loop itself) is an integer, and so it can not decrease indefinitely. This contradiction shows that one will always escape from the table.

4. Let $\mathbf{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$f(x)f(yf(x)) = f(x+y)$$

for all $x, y \in \mathbf{R}^+$.

If $f(x) > 1$ for some x, then for $y = \frac{x}{f(x)-1}$ we get

$$f(x)f(\frac{f(x)x}{f(x)-1}) = f(x)f(yf(x)) = f(x+y) = f(x + \frac{x}{f(x)-1}) = f(\frac{f(x)x}{f(x)-1}),$$

which leads to $f(x) = 1$, a contradiction. Hence $f(x) \leq 1$ for all x. From this we get $f(x+y) = f(x)f(yf(x)) \leq f(x)$ for all x, y, so f is non-increasing.

Suppose there exists an x_0 with $f(x_0) = 1$. Then $f(x_0 + y) = f(y)$ for all y, and then $f(kx_0 + y) = f(y)$ by induction on k. Also, by monotonicity $f(x) = 1$ for all $x \leq x_0$, and since any y can be written as $kx_0 + y_0$ with $y_0 \leq x_0$, we conclude $f(y) = f(y_0) = 1$ for all y.

If there is no x with $f(x) = 1$. Then f is monotone decreasing. Now

$$\begin{aligned} f(x)f(yf(x)) &= f(x+y) = f(yf(x) + x + (1 - f(x))y) \\ &= f(y(f(x)))f((x + (1 - f(x))y)f(y(f(x))), \end{aligned}$$

so $f(x) = f((x + (1 - f(x))y)f(y(f(x)))$. As f is monotone decreasing, this implies $x = x + (1 - f(x))y)f(y(fx)$. Setting $x = 1$, $z = xf(1)$ and $a = f(1)$ we get $f(z) = \frac{1}{1+az}$. Combining this with the case $f(z) = 1$, we conclude that $f(x) = \frac{1}{1+ax}$ for each x with $a \geq 0$. Conversely, a direct verification shows that the functions of this form satisfy the initial equality.

5. A circle is tangent to the continuations of sides CA and CB of the triangle ABC, and is also tangent to the side AB at point P. Prove that the radius of the circle tangent to AP, CP and the circle circumscribed around ABC is equal to the radius of the circle inscribed in ABC.

Let K and M be the tangency points of the circle with AP and CP respectively, L - the point where it touches the circumscribed circle of ABC, T - the middle of the arc AB of the circumscribed circle, I - the center of the inscribed circle of ABC.

The tangent to the circle ABC at T is parallel to the line AB. So the similarity transformation ("stretching") centered at K taking the circle ABC to the circle KLM takes AB to this tangent, and hence T to L. So K, L and T are collinear. We now prove that points K,M and I are collinear. Let M' be the point of intersection of the line KI with the circle KLM. We want to show $M = M'$.

First, let's note that the quadrilateral $LCIM'$ can be inscribed into a circle. In fact, $\angle LCT$ is equal is equal to the half-sum of the arcs LA and AT, which is equal to $\frac{1}{2}(\breve{LA} + \breve{TB}) = \angle LKA$. But LKA is the angle formed by the tangent AK and the chord LK of the circle LKM, and is therefore equal to $LM'K$, subtended by this chord. Hence $\angle LCT = \angle LM'K$ and the quadrilateral $LCMI$ can be inscribed in a circle. Therefore $\angle LM'C = \angle LIC$.

Further, $\angle AKT = \frac{1}{2}(\breve{AT} + \breve{BL}) = \frac{1}{2}(\breve{TB} + \breve{BL}) = \angle LAT$. Hence the triangles TAK and TLA are similar, and so $TK \cdot TL = TA^2$. Also $\angle AIT = \angle CAI + \angle ACI = \angle CAI + \angle TAB = \angle BAI + \angle TAB = \angle TAI$. Therefore TAI is an equilateral triangle, and so $TI^2 = TA^2 = TK \cdot TL$, which implies that the triangles TKI and TIL are similar, and so $\angle LIC = \angle LKI$. Hence $\angle LM'C = \angle LIC = \angle LKM'$, so the angle subtended by the chord LM' is equal to the angle between this chord and $M'C$, so $M'C$ is a tangent, and so $M' = M$.

It is time to use the definition of the point P. Draw a tangent to the inscribed circle of ABC parallel to AB. Let it be tangent to this circle at point F. The circle tangent to the continuations of AC, BC and to AB at P can be obtained from the inscribed circle of ABC by a similarity transformation centered at C. This transformation would take F to P. Hence C, F and P are collinear. Let V be the midpoint of PE, where E is the tangency point of AB with the inscribed circle of ABC. As $IV \parallel FP$ (IV connects the midpoints of the sides of FEP), KIV is similar to KMP, and so $VI = VK$. Hence $IE^2 = VI^2 - VE^2 = VK^2 - VE^2 = (VK + VE)(VK - VE) = EK \cdot PK$.

Now let D be the center of the circle KLM. As $\angle KDP = \angle IKE$, the triangles KDP and EKI are similar. Hence $DK \cdot IE = EK \cdot PK$, and so $IE^2 = EK \cdot PK = DK \cdot IE$, and so $IE = DK$, as wanted.