Berkeley Math Circle
Monthly Contest
Due January 6

1. Find the smallest prime divisor of
$$4^{1001001001} + 1001001001^4.$$

2. Given that
$$x + x^{-1} = \frac{1+\sqrt{5}}{2},$$
find $x^{2000} + x^{-2000}$.

3. Berkeley's professors are holding a New Year's party this winter and n people decide to attend. During the main feast of the celebration the n guests need to sit around a round table – each person sits next to exactly two other people. However, some of the guests hate other people at the party. If person A hates person B then person A and person B cannot sit next to each other at the table for the grand feast. We say the guests have a spite level of m if the list of all pairs (A, B), where person A hates person B, has length m. What is the maximum spite level the group can have and still be guaranteed to find an agreeable seating arrangement for the New Year's feast?

4. Given real $a, b, c, d, e > 0$ such that $a + b + c + d + e = 1$, prove
$$\frac{a^6}{(b^2+d^2+e^2)^2} + \frac{b^6}{(c^2+e^2+a^2)^2} + \frac{c^6}{(d^2+a^2+b^2)^2} + \frac{d^6}{(e^2+b^2+c^2)^2} + \frac{e^6}{(a^2+c^2+d^2)^2} \geq \frac{1}{45}.$$

5. Let r be the inradius and R the circumradius of $\triangle ABC$. Also, let $[ABC]$ denote the area of $\triangle ABC$. Let P, Q, R be the points of tangency of the incircle of $\triangle ABC$ to the sides BC, AC, and AB, respectively, and X be the point of intersection of AP and BQ. If $\frac{R}{r} = 3$ and $[ABC] = 6$ prove that
$$2[AXQ] \cdot [BXR] \cdot [CXP] + [CQX] \cdot [ARX] \cdot [BPX] \leq 2.$$