Berkeley Monthly Contest #2 Solutions

1. (In these solutions, it will be assumed that nobody knows himself.) Let there be n people present at the party. Then there are n different numbers of people that anyone can know: $0, 1, 2, \ldots, n-1$. CASE 1: Suppose that for any a between 0 and $n-1$ someone at the party knows a people. Then there will be someone who knows $n-1$ people, and someone who knows 0 people. However, the person that knows $n-1$ people knows everyone at the party (by definition), and thus also knows the person who knows 0 people. However, knowing is symmetric, thus the person who knows 0 people must also know the person who knows $n-1$ people. This is a contradiction. This leads in to CASE 2: There is some number a between 0 and $n-1$ for which nobody knows exactly a people at the party. Thus there are at most $n-1$ different values of a for which people know exactly a people. However, there are n people; by the pigeonhole principle, that means that there is some b for which two people know excatly b people.

2. Let $a \star b = ab + a + b$. Notice that $a \star b = ab + a + b = (a+1)(b+1) - 1$. Since a and b are symmetric in this expression, this operation is commutative. Also, $(a \star b) \star c = ((a+1)(b+1)-1+1)(c+1)-1 = (a+1)(b+1)(c+1)-1 = (a+1)((b+1)(c+1)+1-1)-1 = a \star (b \star c)$. Thus the operation is associative. Since it is both associative and commutative it can be done in any order and always achieve the same result. Thus the order does not matter. LEMMA: The result after applying $\star$ to the set $\{a_1, a_2, \ldots, a_n\}$ gives $(a_1+1)(a_2+1)\cdots(a_n+1)-1$. <u>Proof:</u> We will prove this using induction. It is true for the $n = 2$ case, since that is the definition of the operation. Suppose that it is true for the $n = k$ case. Then $(a_1 \star a_2 \star \cdots \star a_k) \star a_{k+1} = ((a_1+1)(a_2+1)\cdots(a_k+1)-1) \star a_{k+1} = (((a_1+1)(a_2+1)\cdots(a_k+1)-1)+1)(a_{k+1}+1)-1 = (a_1+1)(a_2+1)\cdots(a_k+1)(a_{k+1}+1)-1$ as was necessary to be shown. Thus the result of applying $\star$ to the set $\{1, 2, \ldots, n\}$ is $(1+1)(2+1)\cdots(n+1)-1 = 1 * 2 * 3 * \cdots * (n+1) - 1 = (n+1)! - 1$. Since $k!$ is even for all $k > 1$, $(n+1)!$ is even for all n greater than 0, and $(n+1)! - 1$ is odd for all such n. Thus the result will always be even if there is more than one number to start off with.

3. LEMMA: (AM-GM *Note: This is well-known, it is not necessary to*

prove it on contests) $\frac{a+b}{2} \geq \sqrt{ab}$ for $a, b \geq 0$ Proof:

$$(a-b)^2 \geq 0 \Rightarrow$$

$$a^2 - 2ab + b^2 \geq 0 \Rightarrow$$

$$a^2 + 2ab + b^2 \geq 4ab \Rightarrow$$

$$(a+b)^2 \geq 4ab \Rightarrow$$

$$\frac{(a+b)^2}{4} \geq ab \Rightarrow$$

Since both a and b are greater than zero, and thus both sides of the expression are greater than zero, it is possible to square root both sides:

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Applying the lemma to a, b, c pairwise, we get

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\frac{a+c}{2} \geq \sqrt{ac}$$

$$\frac{b+c}{2} \geq \sqrt{bc}$$

Multiplying these three expressions we get

$$\frac{a+b}{2}\frac{a+c}{2}\frac{b+c}{2} \geq \sqrt{ab}\sqrt{ac}\sqrt{bc} \Rightarrow$$

$$\frac{(a+b)(b+c)(c+a)}{8} \geq \sqrt{a^2b^2c^2} \Rightarrow$$

$$(a+b)(b+c)(c+a) \geq 8abc.$$

4. LEMMA: In any $\triangle ABC$ with orthocenter H the circumradii of $\triangle ABC$, $\triangle ABH$, $\triangle ACH$ and $\triangle BCH$ are equal. Proof: Let ω be the circumcircle of $\triangle ABC$, and A' the intersection of ray AH with ω. Let H' be the base of the perpendicular from A to BC. Then $\angle A'BC = \angle A'AC$ (because they are based on the same arc of the circle), and $\angle HAC = \angle HBC$ because they are both complementary to $\angle ACB$. And since A, H, A'

are collinear, $\angle A'AC = \angle HAC$. Thus $\angle A'BC = \angle HBC$. Analogously, $\angle A'CB = \angle HCB$. Thus, (since they share side BC) $\triangle HBC \cong \triangle A'BC$. And thus their circumradii are equal. Since $\triangle A'BC$ is inscribed in ω its circumradius is equal to that of $\triangle ABC$, and thus the circumradius of $\triangle HBC$ is equal to the circumradius $\triangle ABC$. The other cases are done analogously.Since G is on the circle with diameter AB $\angle AGB = \frac{\pi}{2}$. Since it is on the circle with diameter AC $\angle AGC = \frac{\pi}{2}$. Because of this $\angle BGC = \angle AGB + \angle AGC = \frac{\pi}{2} + \frac{\pi}{2} = \pi$. Thus G is on segment BC and is the base of the altitude from A onto BC. This makes D the ortocenter of ABC, and by the lemma the circumradii of $\triangle ABD$ and $\triangle ACD$ are equal. Solution 2 (submitted by Philip Sung): Let the centers of the two circles be X and Y, the midpoints of AC and AB, respectively. Since A and G are the two intersection points of the circles, $AG \perp XY$. ALso, since XY bisects AG reflecting A over XY takes it to G. Then G is on BC since XY is the midline of $\triangle ABC$. Also, $AG \perp BC$ since $BC \| XY$. Thus G is the foot of the altitude from A, and D is the orthocenter of the triangle. Reflect C across AG to C'. Then $\triangle DC'A \cong \triangle DCA$, and their circumradii are equal. However, C' is on the circumcircle of $\triangle ABD$ because $\angle AC'D = \angle ACD = \angle ABD$. Thus the circumradii of $\triangle ACD$ and $\triangle ABD$ are equal.It is also possible to solve this using the law of sines.

5. a. There are no domino tiling configurations for a $3 \times n$ rectangle for n odd. In a $3 \times n$ rectangle for n odd, there are an odd number of squares. But clearly any region tiled by 2×1 dominos must consist of an even number of squares. Therefore there are no tilings for a $3 \times n$ rectangle, n odd.b. There are three 3×2 rectangles. The domino covering the upper left square can be either vertical or horizontal. If it is vertical, this forces the domino in the upper right corner to be verical as well, fixing the position (horizontal, bottom) of the third domino. If it is horizontal the two remaining dominos may fill the 2×2 square below either horizontally or vertically. This gives rise to three domino configurations as desired.c. There are two $3 \times n$ rectangles, for n even, that cannot be split into smaller $3 \times n$ rectangles. For each $k < n$ we can imagine a vertical line seperating the rectangle in a $3 \times k$ portion and a $3 \times (n-k)$ portion. There must be a domino spanning this boundry for each k line or else we would be able to split the rectangle into two smaller ones (condtradiction hypothesis). For a domino to

span a vertical line it must be horizontal. As the rectangle has a height of 3 there can be only $1, 2$, or 3 spanning this line. If there are 1 or 3 rectangles spanning the line then on either side we will have some number of whole dominos, each covering an even number of squares an an odd number of "half" dominos covering one square on each side of the line. This gives an odd number of squares on each side of the line. Similarly with 2 rectangles spanning the line we have an even number of squares on either side of the line. Thus if k is even we must have 2 dominos spanning our line and if k is odd 1 or 3. But notice that if at each even junction we have exactly 2 dominos spanning the boundry, it must the same two rows at each junction, and the third row must have its boundary along this line. Our only choice left then is which of the three rows has its boundry at the even intervals. Notice that if it is the middle row, then when we reach either end we are left with 4 fragmented squares at the corners; thus this configuration is impossible. This leaves us with two possible configurations. Finally we notice these possibilites yeild actual configurations. The 4 leftover squares here, form two contiguous chuncks in which we can place a domino. Therefore: for n even, there are two 3xn retangles that cannot be split into smaller $3 \times n$ rectangles.d. We can form a recurssion for the total number of $3 \times 2n$ rectangles (a_n) by noticing that each legal rectangle consists of some indivisible $3 \times 2k$ rectangle at the leftmost end and then a $3 \times 2(n-k)$ rectangle for the rest of it. As there are two such $3 \times 2k$ rectangles (3 if $k = 1$) we get the recursion. $A_n = 3 * A_{n-1} + 2 * (A_{n-2} + \ldots + A_0)$, letting $A_0 = 1$. If we comibine the recusion for a_n and a_{n-1} $(an - 1 = 3 * an - 2 + \ldots)$ we get $a_n = 4a_{n-1} - an - 2$.e. Now, to construct the closed form for the formula we can assume that it is a geometric series. That gives us the equation $r^2 = 4r - 1$ (if r is the ratio between succesive terms). SOlving this, we get that $r = 2 \pm \sqrt{3}$. Then $a_n = X(2 + sqrt3)^n + Y(2 - sqrt3)^n$. Since $a_0 = 1$ and $a_1 = 3$ by solving the equation we get that $X = \frac{3+sqrt3}{6}$, $Y = \frac{3-sqrt3}{6}$, so the closed form for the equation is

$$a_n = \frac{3 + sqrt3}{6}(2 + sqrt3)^n + \frac{3 - sqrt3}{6}(2 - sqrt3)^n.$$