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Berkeley Math Circle Monthly Contest #3
Solutions

The complete set of functions described by the functional equations is the set of constant functions $f :\to$. Letting $x = y$, we have: $f(x) + f(x) = f(\frac{x+x}{2}) + f(3x)$; which implies $f(x) = f(3x)$ for any integer x. Applying this to the original functional equation and letting $y = 0$ and $x = 2n$, $n \in$, we get: $f(0) = f(\frac{2n+0}{2})$. Thus we have for any integer n, and any function f satisfying the original functional equation, $f(n) = f(0)$; that is, f is a constant function. It remains to be verified that any such constant function satisfies the original functional equation. And, indeed $f(0) + f(0) = f(0) + f(0)$.

The minimum value of the expression is $\frac{3}{2}$. By the Cauchy-Schwartz inequality,

$$(\frac{a^2}{ab+ac} + \frac{b^2}{bc+ba} + \frac{c^2}{ca+cb})((ab+ac) + (bc+ba) + (ca+cb)) \geq (a+b+c)^2$$

But, by the rearrangement inequality, $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca)$ Combining and dividing, we get that:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3(ab+bc+ca)}{2(ab+bc+ca)} = \frac{3}{2}$$

All the remains to be shown is that the expression can in fact attain $\frac{3}{2}$. Taking $a = b = c$, we have $\frac{a}{a+a} + \frac{a}{a+a} + \frac{a}{a+a} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$, Completing the proof.

For the purposes of this proof we number the goblins from 1 to N, in order of increasing vileness (i.e. if $i > j$ then goblin i is more vile than goblin j)

Which goblins are killed for N goblins starting For $N < 51$: No goblins killed2: Goblin 2 killed3: Goblin 1 killed4: Goblins 1,3,4 killed5: Goblins 2,4 killedFor $N \geq 6N \equiv 0(3)$: Goblins $2, \ldots, N-1$ are killed$N \equiv 1(3)$: Goblins $1, \ldots, N-1$ are killed$N \equiv 2(3)$: Goblins $\lfloor N/2 \rfloor + 1), \ldots, N-1$ are killed

Proof: Let $P(N)$ be the plan finally accepted by N goblins.We should note that we can $P(N+1)$ if we know $P(N)$. If the goblins reject a plan then the goblins killed if there had been N goblins are killed along with goblin $N+1$. Note that a goblin will only vote for $P(N+1)$ if it prefers $P(N+1)$ to $P(N)$. If at least $\lfloor (N/2) \rfloor + 1$ prefer $P(N+1)$ to $P(N)$ then the plan is accepted. If no plans are accepted then $P(N+1) = P(N) \bigcup N+1$. Otherwise, from the set of all plans that might be accepted in group of $N+1$, $P(N+1$ is the one that goblin $N+1$ prefers.

We see that the most vile goblin has two ways he can convince people to vote his plan. He can either pick a plan that has more vile goblins killed then $P(N)$ and not kill too many goblins that would other wise be spared (at most $\lceil \frac{N+1}{2} \rceil - 1$ goblins), or he can spare enough goblins lives that would otherwise be killed (at least $\lfloor N/2 \rfloor + 1$ including goblin $N+1$). I will use set $(a, b, c, \ldots)$ to denote which goblins are killed in a given plan.

Cases $N = 1$ through $N = 6$ are left as an exercise for the reader. Be careful: the logic around $N = 4$ is tricky.

Suppose $N \equiv 1$ (3).Clearly there is no better plan for the vilest goblin then to kill all his competitors and live himself. Thus it suffices to show this plan will be accepted. This plan kills $N-1$ goblins, all the goblins but the Nth. If the plan is not approved both goblin 1 and goblin $N-1$ survive, and $N-2$ goblins are killed. Clearly then, all the goblins 2 through $N-2$, who will die regardless of whether or not the plan is approved, prefer to kill more goblins (by axiom 2) and vote to approve the plan. In addition, the vilest goblin will die if the plan is not approved and so vote for its approval as well. Thus all but two goblins vote for the plan, a majority for $N \geq 6$, and the plan is approved.

Suppose $N \equiv 2$ (3).We see that there is no way the vilest Goblin can kill more than the $N-1$ goblins that will die if his plan is not approved unless he kills everyone. If he kills everyone, he dies and as we will see this is unnecessary. Thus the vilest Goblin must save goblins that would otherwise be killed. We see that all the goblins except goblin $N-1$ will die if the plan is not approved; thus for every goblin he spares, except goblin $N-1$ and including himself, he gets one vote of approval. The goblin $N-1$ will never vote for the plan. So, to insure victory the goblin must spare at least $\lfloor \frac{N}{2} + 1$ goblins. To accomplish this, and not kill himself, and kill the highest ranking goblins possible he kills goblins $\lfloor \frac{N}{2} \rfloor + 1$ through $N-1$ inclusive.

Suppose $N \equiv 0$ (3).If the vilest goblin selects all the goblins but himself and goblin 1, the plan will pass. The plan clearly kills more goblins than a rejection and goblins $\lfloor \frac{N-1}{2} + 1$ through $N-2$ will die anyway and thus vote to approve this plan. With the added vote of the vilest goblin and goblin 1 (both spared), this plan is approved. Further, the only plan more beneficial to the vilest goblin is to kill everyone but himself. This plan would be rejected, since then the same number of goblins would be killed (after killing goblin N), but more vile goblins would be killed than by another plan.

figure[h]

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